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A THEOREM OF USE IN WAVE THEORY

By Edmund Pinney

1. Introduction. The purpose of this paper is to prove a theorem which has been of considerable value in reducing the solutions of some problems in the theory of elastic waves to tractable form susceptible to numerical calculation. It seems worthwhile to present it here in view of its applicability to other problems in this theory as well as possible applications in other theories involving solutions of the wave equation. The theorem can be applied to regimes in which one or more wave velocities appear. In the theory of elastic waves there are two such wave velocities corresponding to irrotational and equi-voluminar waves respectively. It may have applications in the optical theory of crystals.

The theorem is applied to finding expressions for surface displacements due to a P-wave point source in a semi-infinite medium.

2. Theorem. Let $p_1 > 0, p_2 > 0, \cdots, p_n > 0, r > 0, 0 < k_1 < k_2 < \cdots < k_n$. Let t be real, and s complex. Define

$$\lambda_m \equiv \lambda_m(s) = \sqrt{(s^2 - k_m^2)}, \qquad m = 1, 2, \cdots, n \tag{2.1}$$

the s-plane being cut from $-k_n$ to $+k_n$ and the sheet of the resulting Riemann surface being chosen for which $\lambda_m(s) \to s$ as $s \to \infty$, $m = 1, 2, \cdots, n$.

Let $G(s)$ be a function of the complex variable s such that

(i) $G(s)$ is meromorphic for $\mathrm{Re}(s) \geqq 0$ outside the interval $0 \leqq \mathrm{Re}(s) \leqq k_n$, $\mathrm{Im}(s) = 0$;

(ii) the poles of $G(s)$ lie to the right of the curves

$$\mathrm{Re}\,[\textstyle\sum_1^n p_m\lambda_m(s) \pm irs] = 0; \tag{2.2}$$

(iii) $G(s) - A = o(s^\nu)$ as $s \to \infty$, $\mathrm{Re}(s) \geqq 0$ for some constant A and some $\nu < 0$;

(iv) $u^{-1/2}G(u \pm i0)$ is continuous of bounded variation for $0 \leqq u < k_1$.

Let C be a contour in the first and fourth quadrants of the s-plane, beginning at $0 - i0$, surrounding the interval $0 \leqq \mathrm{Re}(s) \leqq k_n$, $\mathrm{Im}(s) = 0$, and the poles of $G(s)$ in the counter-clockwise direction and returning to $0 + i0$. Let C lie entirely to the right of curves (2.2) except for segments lying on the real axis.

Define

$$\begin{aligned} h(u) &= \textstyle\sum_1^n p_m\sqrt{(k_m^2 - u^2)}, \\ j(v) &= \textstyle\sum_1^n p_m\sqrt{(k_m^2 + v^2)}, \\ \mu_m(u) &= \sqrt{(k_m^2 - u^2)}, \qquad m = 1, 2, \cdots, n. \end{aligned} \tag{2.3}$$

Let u_1 denote the real root of $h'(u) + r = 0$, $(0 < u_1 < k_1)$ and define

$$\delta = h(u_1) + ru_1 \tag{2.4}$$

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Then if U_1 is the smallest positive root of $h(u) + ru = |t|$ when $\sum_1^n p_m k_m < |t| < \delta$ and $U = 0$ when $|t| < \sum_1^n p_m k_m$, and if

$$H(t) \equiv H(p_1, p_2, \cdots, p_n, r, t) = \int_0^\infty \int_C J_0(\omega r s) \exp\left[-\omega \sum_1^n p_m \lambda_m(s)\right] G(s) e^{i\omega t}\, ds\, d\omega, \tag{2.5}$$

we have

$$H(p_1, p_2, \cdots, p_n, r, t) = \frac{\pi i A}{\sqrt{(\sum_1^n p_m)^2 + r^2}} - 2i \int_0^U \frac{G(u + i0 \operatorname{sgn} t)\, du}{\sqrt{[|t| - h(u)]^2 - r^2 u^2}} - \int_0^\infty \frac{G(iv)\, dv}{\sqrt{[t - j(v)]^2 + r^2 v^2}} + \int_0^\infty \frac{G(-iv)\, dv}{\sqrt{[t + j(v)]^2 + r^2 v^2}} \tag{2.6}$$

whenever $t^2 < \delta^2$, and

$$\begin{aligned} H(p_1, p_2, \cdots, p_n, r, t) &= i \operatorname{sgn} t \int_0^{k_1} \left\{ -\frac{G(u + i0)}{\sqrt{[t - h(u)]^2 - r^2 u^2}} + \frac{G(u - i0)}{\sqrt{[t + h(u)]^2 - r^2 u^2}} \right\} du \\ &\quad - \sum_1^{n-1} q \int_{k_q}^{k_{q+1}} \frac{G(u - i0)\, du}{\sqrt{[\sum_1^q p_m \lambda_m(u) + i \sum_{q+1}^n p_m \mu_m(u) - it]^2 + r^2 u^2}} \\ &\quad + \sum_1^{n-1} q \int_{k_q}^{k_{q+1}} \frac{G(u - i0)\, du}{\sqrt{[\sum_1^q p_m \lambda_m(u) - i \sum_{q+1}^n p_m \mu_m(u) - it]^2 + r^2 u^2}} \\ &\quad + 2\pi i \sum Res \frac{G(s)}{\{[\sum_1^n p_m \lambda_m(s) - it]^2 + r^2 s^2\}^{1/2}} \end{aligned} \tag{2.7}$$

whenever $t^2 > \delta^2$. The arguments of the radicals in the denominators of the third, fourth, and fifth terms in (2.7) lie between 0 and $\pi/2$, $-\pi/2$ and 0, and $-\pi/2$ and $\pi/2$, respectively. The residue summation is over the poles of $G(s)$.

3. Preliminary lemmas. Before proving the theorem we must prove two lemmas.

LEMMA 1. If $u^{-1/2} g(u)$ is continuous of bounded variation for $0 \leqq u < k_1$, then for $t \neq \mp\delta$,

$$\lim_{\epsilon \to 0} \int_0^\infty e^{i\omega t} \int_U^{U+\epsilon} g(u) J_0(\omega r u) \exp\left[\pm i\omega h(u)\right] du\, d\omega = 0 \tag{3.1}$$

for $0 \leqq U < k_1$, and where $\epsilon \to 0$ from above when $U = 0$.

PROOF. We may write[1]

$$J_0(x) = \sqrt{2/\pi x} \cos(x - \pi/4) + R(x) \tag{3.2}$$

where

$$|R(x)| < \tfrac{1}{2} x^{-3/2} \quad \text{for } x > 1. \tag{3.3}$$

[1] G. N. Watson, "Bessel Functions," Macmillan, 1944, §§7.1, 7.32.

Accordingly $| J_0(x) | < \frac{3}{2}x^{-1/2}$ for $x > 1$. Then

$$| J_0(x) | < 3/\sqrt{2(x+1)} \quad \text{for} \quad x \geqq 0. \tag{3.4}$$

That this is true for $x > 1$ follows from the above and the fact that $(x+1)/x < 2$ for $x > 1$. It also holds when $0 \leqq x \leqq 1$ because $| J_0(x) | \leqq 1$ for all real x.

Now[2] for any finite $\tau_0 > 0$

$$\begin{aligned}
&\lim_{\epsilon \to 0} \left| \int_0^{\tau_0/|\epsilon|} e^{i\omega t} \int_U^{U+\epsilon} g(u) J_0(\omega r u) \exp [\pm i\omega h(u)]\, du\, d\omega \right| \\
&= \lim_{\epsilon \to 0} \left| \int_U^{U+\epsilon} g(u) \int_0^{\tau_0/|\epsilon|} J_0(\omega r u) \exp \{i\omega[t \pm h(u)]\}\, d\omega\, du \right| \\
&\leqq \lim_{\epsilon \to 0} \int_U^{U+\epsilon} | g(u) | \int_0^{\tau_0/|\epsilon|} | J_0(\omega r u) |\, d\omega\, du \\
&< \frac{3}{\sqrt{2}} \lim_{\epsilon \to 0} \int_U^{U+\epsilon} | g(u) | \int_0^{\tau_0/|\epsilon|} (\omega r u + 1)^{-1/2}\, d\omega\, du \\
&= \frac{3\sqrt{2}}{r} \lim_{\epsilon \to 0} | \epsilon |^{-1/2} \int_U^{U+\epsilon} \frac{| g(u) |}{u} [(u r \tau_0 + | \epsilon |)^{1/2} - | \epsilon |^{1/2}]\, du \\
&= \frac{3\sqrt{2}}{r} \lim_{\epsilon \to 0} | \epsilon |^{1/2} \int_0^1 \frac{| g(U + \sigma\epsilon) |}{U + \sigma\epsilon} [\{(U + \sigma\epsilon) r \tau_0 + | \epsilon |\}^{1/2} - | \epsilon |^{1/2}]\, d\sigma \\
&= 0
\end{aligned}$$

because the integral is bounded. It follows that the lower limit of the outside integral in (3.1) may be replaced by $\tau_0/| \epsilon |$ without changing the limit.

Let $u = U + \sigma\epsilon$, $\omega = \tau/| \epsilon |$ in (3.1), after making the indicated change in the lower limit of the outer integral. Then the left side of (3.1) equals

$$\lim_{\epsilon \to 0} \int_{\tau_0}^{\infty} \exp [i\tau \{t \pm h(U)\}/| \epsilon |] A(\tau)\, d\tau \tag{3.5}$$

where

$$A(\tau) = \int_0^1 g(U + \sigma\epsilon)\, e^{\pm i\tau f(\sigma)} J_0(\tau U r/\epsilon + \tau\sigma r)\, d\sigma, \tag{3.6}$$

$$f(\sigma) = [h(U + \sigma\epsilon) - h(U)]/| \epsilon |. \tag{3.7}$$

When $U = 0$ the lower limit of integration in (3.6) may be replaced by $1/(\tau r)$. To see this, note that for ϵ sufficiently small there exists a $K > 0$ such that $| g(u) | < K u^{1/2}$ when $0 \leq u \leq \epsilon$, $\epsilon > 0$. Therefore

$$\left| \int_0^{1/(\tau r)} g(\sigma\epsilon) e^{\pm i\tau f(\sigma)} J_0(\tau\sigma r) \right| d\sigma \leq K\epsilon^{1/2} \int_0^{1/(\tau r)} \sigma^{1/2}\, d\sigma = \tfrac{2}{3} K\epsilon^{1/2} (\tau r)^{-3/2}.$$

Taking the absolute value of (3.5) and inserting this in place of $| A(\tau) |$, the resulting expression approaches zero as $\epsilon \to 0$. Therefore the $0 \leq \sigma \leq 1/(\tau r)$ part of the integral of (3.6) has a zero contribution to (3.5).

[2] E. C. Titchmarsh, "Theory of Functions," Oxford, 1932, §12.6.

It follows that the argument of the Bessel function may always be taken as ≥ 1. Accordingly, we may replace the Bessel function by the first term on the right of (3.2). To see this we note that the contribution to (3.6) of the second term on the right of (3.2) is, by (3.3), $0(|\epsilon|^{3/2}\tau^{-3/2})$ when $U \neq 0$, and $0(\epsilon^{1/2}\tau^{-3/2}ln\tau)$ when $U = 0$, the latter result involving the fact that $|g(u)| < Ku^{1/2}$ when $0 \leq u \leq \epsilon$, and that the lower limit in (3.6) may be replaced by $1/(\tau r)$. Taking the absolute value of (3.5), and inserting this term in place of $A(\tau)$, the resulting expression approaches zero as $\epsilon \to 0$. Therefore the second term on the right of (3.2) has a zero contribution to (3.5).

Accordingly $A(\tau)$ in (3.6) may be replaced by

$$\sqrt{\frac{2|\epsilon|}{\pi\tau r}} \int_p^1 \frac{g(U+\sigma\epsilon)}{\sqrt{U+\sigma\epsilon}} e^{\pm ir f(\sigma)} \cos(\tau U r/\epsilon + \tau\sigma r - \pi/4)\, d\sigma$$

where

$$p = 1/(\tau r) \quad \text{when} \quad U = 0, \qquad p = 0 \quad \text{when} \quad U \neq 0. \tag{3.8}$$

Therefore (3.5) may be written

$$\lim_{\epsilon\to 0} \int_{\tau_0}^{\infty} \sqrt{\frac{|\epsilon|}{2\pi\tau r}} \{e^{ir\alpha/|\epsilon| + i\pi/4} C(\tau, r) + e^{ir\beta/|\epsilon| + i\pi/4} C(\tau, -r)\}\, d\tau \tag{3.9}$$

where

$$\alpha = t \pm h(U) + rU, \qquad \beta = t \pm h(U) - rU \tag{3.10}$$

$$C(\tau, r) = \int_p^1 \frac{g(U+\sigma\epsilon)}{\sqrt{U+\sigma\epsilon}} e^{ir[\pm f(\sigma)+\tau\sigma]}\, d\sigma. \tag{3.11}$$

Now assume that $U \neq u_1$. Then, for ϵ sufficiently small, $\pm f'(\sigma) + r \neq 0$ for $0 \leq \sigma \leq 1$. The functions

$$\frac{g(U+\sigma\epsilon)}{\sqrt{U \pm \sigma\epsilon}[\pm f'(\sigma) + r]}, \qquad e^{ir[\pm f(\sigma)+\tau\sigma]}$$

are continuous and of bounded variation over the interval $0 \leq \sigma \leq 1$. Therefore[3] (3.11) may be integrated by parts, giving

$$\begin{aligned} C(\tau, r) = \frac{1}{ir}\Bigg\{ & \frac{g(U+\epsilon)}{\sqrt{U+\epsilon}[\pm f'(1) + r]} e^{ir[\pm f(1)+\tau]} \\ & - \frac{g(U+p\epsilon)}{\sqrt{U+p\epsilon}[\pm f'(p) + r]} e^{ir[\pm f(p)+pr]} \\ & - \int_p^1 \frac{d}{d\sigma}\left[\frac{g(U+\sigma\epsilon)}{\sqrt{U+\sigma\epsilon}[\pm f'(\sigma) + r]}\right] e^{ir[\pm f(\sigma)+\tau\sigma]}\, d\sigma \Bigg\}. \end{aligned}$$

For ϵ sufficiently small, this integral has a bound independent of τ, and similarly

[3] S. Saks, "Theory of the Integral," Warsaw, 1937 (14.1).

for $C(\tau, -r)$. Therefore $|C(\tau, r)| < K'/\tau$, $|C(\tau, -r)| < K'/\tau$ for some K' independent of τ. The absolute value of the integral in (3.9) is therefore less than

$$\int_{\tau_0}^{\infty} \sqrt{2|\epsilon|/\pi\tau}\, K'\tau^{-3/2}\, d\tau$$

which approaches zero as $\epsilon \to 0$. This proves (3.1) for $U \not\equiv u_1$.

Now suppose $U = u_1$, and consider only the lower sign in (3.1). The analysis below (3.11) still applies to $C(\tau, r)$ but no longer applies to $C(\tau, -r)$. Accordingly (3.9) becomes

$$\frac{e^{i\pi/4}}{\sqrt{2\pi\tau}} \lim_{\epsilon\to 0} |\epsilon|^{1/2} \int_{\tau_0}^{\infty} e^{i\tau\beta/|\epsilon|}\, C(\tau, -r)\tau^{-1/2}\, d\tau.$$

Since $t \not\equiv \delta$, by (2.4), (3.10), $\beta \not\equiv 0$. Integrating by parts,[4] this expression becomes

$$\frac{e^{-i\pi/4}}{\sqrt{2\pi\tau}} \beta^{-1} \lim_{\epsilon\to 0} |\epsilon|^{3/2} \left\{ - e^{i\tau\beta/|\epsilon|}\, C(\tau_0, -r)\tau_0^{-1/2} \right.$$
$$\left. + \frac{1}{2}\int_{\tau_0}^{\infty} e^{i\tau\beta/|\epsilon|}\, C(\tau, -r)\tau^{-3/2}\, d\tau - \int_{\tau_0}^{\infty} e^{i\tau\beta/|\epsilon|}\, C_\tau(\tau, -r)\tau^{-1/2}\, d\tau \right\}.$$

Since $C(\tau, -r)$ is bounded, the first two terms in the bracket are bounded and have a zero contribution to the limit. It remains to prove that

$$\lim_{\epsilon\to 0} |\epsilon|^{3/2} \int_{\tau_0}^{\infty} e^{i\tau\beta/|\epsilon|}\, C_\tau(\tau, -r)\tau^{-1/2}\, d\tau = 0. \tag{3.12}$$

By (3.11),

$$C_\tau(\tau, -r) = -i \int_p^1 \frac{g(U + \epsilon\sigma)}{\sqrt{U + \epsilon\sigma}} [f(\sigma) + r\sigma] e^{-i\tau[f(\sigma)+r\sigma]}\, d\sigma. \tag{3.13}$$

The functions

$$\frac{g(U + \sigma\epsilon)}{\sqrt{U + \sigma\epsilon}} \frac{f(\sigma) + r\sigma}{f'(\sigma) + r}, \qquad e^{-i\tau[f(\sigma)+r\sigma]}$$

are continuous and of bounded variation over the interval $0 \le \sigma \le 1$. Therefore[3] (3.13) may be integrated by parts, giving

$$C_\tau(\tau, -r) = \frac{1}{\tau}\left\{ \frac{g(U + \epsilon)}{\sqrt{U + \epsilon}} \frac{f(1) + r}{f'(1) + r}\, e^{-i\tau[f(1)+r]} \right.$$
$$- \frac{g(U + p\epsilon)}{\sqrt{U + p\epsilon}} \frac{f(p) + rp}{f'(p) + r}\, e^{-i\tau[f(p)+rp]}$$
$$\left. - \int_p^1 \frac{d}{d\sigma}\left[\frac{g(U + \sigma\epsilon)}{\sqrt{U + \sigma\epsilon}} \frac{f(\sigma) + r\sigma}{f'(\sigma) + r} \right] e^{-i\tau[f(\sigma)+r\sigma]}\, d\sigma \right\}.$$

For ϵ sufficiently small this integral has a bound independent of τ, so there exists a constant K'', independent of τ, such that $|C_\tau(\tau, -r)| < K''/\tau$. Taking

[4] E. C. Titchmarsh, loc. cit., §§12.11, 1.88.

the absolute value of the integral in (3.12) and inserting this, it is clear that (3.12) is true. (3.1) is therefore established for the lower sign where $U = u_1$. The proof for the upper sign is similar.

LEMMA 2. The integral

$$\int^{\infty} J_0(\omega a)e^{i\omega b}\,d\omega$$

is uniformly convergent in a, b for a, b real, $a > 0$, $a \neq b$, $a \neq -b$.

PROOF. By (3.2), (3.3), for $L > 0$,

$$\int_L^{\infty} J_0(\omega a)e^{i\omega b}\,d\omega = \frac{1}{\sqrt{2\pi a}}\left[\int_L^{\infty} e^{i\omega(b+a)-\pi i/4}\,\omega^{-1/2}\,d\omega + \int_L^{\infty} e^{i\omega(b-a)+\pi i/4}\,\omega^{-1/2}\,d\omega\right] + \int_L^{\infty} R(\omega a)e^{i\omega b}\,d\omega.$$

For $La > 1$,

$$\left|\int_L^{\infty} R(\omega a)e^{i\omega b}\,d\omega\right| < \tfrac{1}{2}a^{-3/2}\int_L^{\infty}\omega^{-3/2}\,d\omega = a^{-3/2}L^{-1/2}.$$

Also, the first two integrals in the above expression may be evaluated in terms of Fresnel's integrals.[5] In fact

$$\int_L^{\infty} J_0(\omega a)e^{i\omega b}\,d\omega = \frac{e^{-\pi i/4}}{\sqrt{a\,|\,b+a\,|}}\,F[L(b+a)] + \frac{e^{\pi i/4}}{\sqrt{a\,|\,b-a\,|}}\,F[L(b-a)] + \eta, \tag{3.14}$$

where

$$F[x] = \tfrac{1}{2} - C(|\,x\,|) + i\,\operatorname{sgn} x[\tfrac{1}{2} - S(|\,x\,|)]$$

and where, for $La > 1$,

$$|\,\eta\,| < a^{-3/2}L^{-1/2}.$$

For a outside arbitrarily small fixed intervals about 0, b, $-b$, the right side of (3.14) approaches zero uniformly as $L \to \infty$, so the lemma is proved.

4. Proof of the Theorem. Let $C = C_1 + C_2 + C_3$, where for some u_2, $u_1 < u_2 < k_1$, C_1 is a straight line along the lower side of the real axis from $s = 0 - i0$ to $s = u_2 - i0$, C_2 is a straight line along the upper side of the real axis from $s = u_2 + i0$ to $s = 0 + i0$, and C_3 is a contour surrounding the interval $u_2 < \operatorname{Re}(s) < k_n$, $\operatorname{Im}(s) = 0$, and the poles of $G(s)$ in the counterclockwise direction, but lying wholly to the right of the curves (2.2). Let C' be the broken contour obtained by removing from C_1 and C_2 arbitrarily small intervals about $s = 0$ and about the points for which $h(u) \pm ru + t = 0$ if such points exist.

[5] E. Jahnke, and F. Ende, "Tables of Functions," Dover, 1945, pp. 35, 36.

By Lemma 1, when $t \neq \pm\delta$, the right side of (2.5) differs by an arbitrarily small amount from the corresponding expression in which C is replaced by C'. In the latter expression the order of integration may be changed for, by Lemma 2, the resulting ω-integral is uniformly convergent on the part of C' for which the λ_m functions are pure imaginary; it is a-fortiori so on the part of C' where they are complex. Therefore, going to the limit as $C' \to C$, the order of integration on the right of (2.5) may be interchanged, giving

$$H(t) = \int_C G(s)F(s, t)\, ds, \tag{4.1}$$

where

$$F(s, t) = \int_0^\infty J_0(\omega r s) \exp\left[-\omega \textstyle\sum_1^n p_m \lambda_m(s)\right] e^{i\omega t}\, d\omega. \tag{4.2}$$

$F(s, t)$ has branch points in the s-plane on the curves (2.2) at points corresponding to

$$t = -i \textstyle\sum_1^n p_m \lambda_m(s) \pm rs. \tag{4.3}$$

The curves (2.2) are of the form shown in Fig. 1. The points on the curve AU correspond to the upper sign in (4.3) and to t varying from δ to ∞. The points on the curve LA' correspond to the lower sign in (4.3) and to t varying from $-\infty$ to $-\delta$.

$F(s, t)$ is defined by (13) only to the right of the curves (2.2). In fact, in this region[6]

$$F(s, t) = \{[\textstyle\sum_1^n p_m\lambda_m(s) - it]^2 + r^2s^2\}^{-1/2}, \tag{4.4}$$

where the value of the square root is taken for which

$$|\textstyle\sum_1^n p_m\lambda_m(s) - it + [F(s, t)]^{-1}| > |rs|. \tag{4.5}$$

Cutting the s-plane by connecting together the branch points of (4.4) in an appropriate manner, (4.4) may be employed to continue analytically the function $F(s, t)$ throughout the region of analyticity of (4.4). It is to be noted that the condition (4.5) need not apply in the region of continuation. Corresponding to each fixed t, the branch points of (4.4) are points on the curves (2.2) illustrated in Fig. 1.

Suppose $-\delta < t < \delta$. Then $G(s)F(s, t)$ has no branch points outside the contour C except at $s = U$. Since $F(s, t) \to [(\sum_1^n {}_m p_m)^2 + r^2]^{-1/2}/s$ as $s \to \infty$, by iii), C may be deformed to the imaginary axis plus the right-hand semicircle at infinity, plus a clockwise contour from $s = 0 + i0$ about $s = U$ to $s = 0 - i0$, so that

$$H(t) = -i \int_{-\infty}^{\infty} G(iv)F(iv, t)\, dv + \pi i A[\textstyle\sum_1^n p_m)^2 + r^2]^{-1/2} - 2i \int_0^U \frac{G(u + i0 \operatorname{sgn} t)\, du}{\sqrt{[\,|t| - h(u)]^2 - r^2u^2}}. \tag{4.6}$$

[6] G. N. Watson, loc. cit., 13.2 (1).

Suppose $t > \delta$ or $t < -\delta$. Then C may be shrunk so that it just surrounds the interval $0 < \mathrm{Re}(s) < k_n$, $\mathrm{Im}(s) = 0$, giving

$$H(t) = \int_0^{k_n} G(u - i0)F(u - i0, t)\, du - \int_0^{k_n} G(u + i0)F(u + i0, t)\, du + 2\pi i R, \tag{4.7}$$

where R is the sum of the residues of $G(s)F(s, t)$ at the poles of $G(s)$.

Substituting from (4.4) the appropriate values of $F(s, t)$, (4.6), (4.7) give (2.6), (2.7).

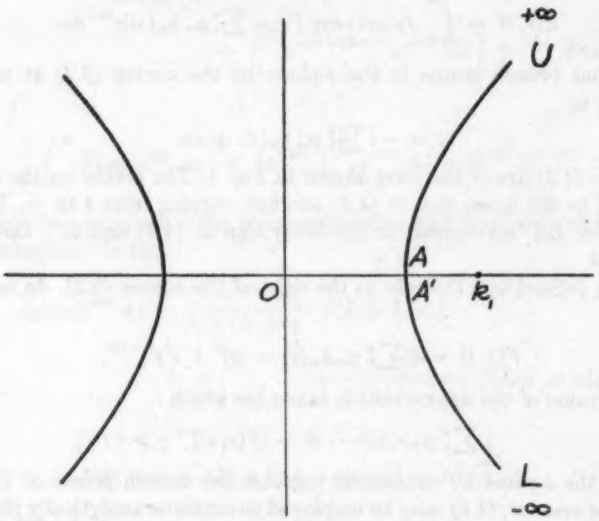

FIGURE 1

5. An application to the theory of elastic waves. In an x, y, z-rectangular coordinate system, suppose the region $z > 0$ is filled with an elastic medium having P- and S-wave velocities c_p and c_s respectively. Suppose a small spherical volume V is instantaneously injected at the point $(0, 0, a)$. Let $R(r, t)$, and $Z(r, t)$ be the displacements of points on the surface $z = 0$ radially outward from the origin $(0, 0, 0)$ and in the $-z$ direction (i.e., outward from the material), respectively, where $r = \sqrt{(x^2 + y^2)}$. By a rather long but straightforward calculation which will appear in another paper,

$$R(r, t) = \frac{V}{2\pi(1 - k^2)} \frac{r}{(a^2 + r^2)^{3/2}} - \frac{V}{4\pi^2 i} \frac{\partial}{\partial r} \int_0^\infty \int_C J_0(\omega r s) e^{-i\omega\lambda_1(s)} \frac{\lambda_1(s)\, ds \cos \omega t\, d\omega}{(s^2 - \frac{1}{2})^2 - s^2\lambda_2(s)\lambda_1(s)} \tag{5.1}$$

$$Z(r,t) = \frac{V}{2\pi(1-k^2)}\frac{a}{(a^2+r^2)^{3/2}} - \frac{V}{4\pi^2 i}\frac{\partial}{\partial a}\int_0^\infty\int_C J_0(\omega r s)e^{-\omega\lambda_2(s)}\frac{(s^2-\frac{1}{2})[\lambda_2(s)]^{-1}\,s\,ds\cos\omega t\,d\omega}{(s^2-\frac{1}{2})^2 - s^2\lambda_2(s)\lambda_1(s)} \tag{5.2}$$

where $k = c_s/c_p$, $\lambda_2(s) = \sqrt{(s^2-k^2)}$, $\lambda_1(s) = \sqrt{(s^2-1)}$, where the contour C begins at $s = 0 - i0$, surrounds the points $s = k$, $s = 1$, and the poles of the integrand in the counterclockwise direction, and ends at $s = 0 + i0$, and where the units have been taken so that $c_s = 1$.

Hitherto most of the investigations of elastic waves in semi-infinite media have obtained final expressions substantially similar to (5.1), (5.2). However these expressions are poorly adapted to either analytic treatment or numerical analysis. We may apply the theorem of §2 to reduce the double integrals.

The details of the calculation will be given for the integral in (5.1). Comparing with (2.5),

$$R(r,t) = \frac{V}{2\pi(1-k^2)}\frac{r}{(a^2+r^2)^{3/2}} - \frac{V}{8\pi^2 i}\frac{\partial}{\partial r}[H(a,r,t) + H(a,r,-t)], \tag{5.3}$$

where $n = 2$, $k_1 = k < 1$, $k_2 = 1$, $p_1 = a$, $p_2 = 0$, and

$$G(s) = \frac{\lambda_1(s)s}{(s^2-\frac{1}{2})^2 - s^2\lambda_2(s)\lambda_1(s)}. \tag{5.4}$$

$G(s)$ has a simple pole at $s = \sqrt{\eta_0}$, where η_0 is the real root of

$$(1-k^2)\eta^3 - (\tfrac{3}{2}-k^2)\eta^2 + \tfrac{1}{2}\eta - \tfrac{1}{16} = 0. \tag{5.5}$$

Conditions i) – iv) of the Theorem are clearly satisfied with $A = -2/(1-k^2)$.

By (2.3), (2.4), $u_1 = kr/D$, $\delta = kD$, where

$$D = (a^2+r^2)^{1/2}. \tag{5.6}$$

By (5.4),

$$G(\pm iv) = \frac{-v\sqrt{(v^2+1)}}{(v^2+\frac{1}{2})^2 - v^2\sqrt{(v^2+k^2)}\sqrt{(v^2+1)}}$$

$$G(u \pm i0) = \frac{\pm iu\sqrt{(1-u^2)}}{(u^2-\frac{1}{2})^2 + u^2\sqrt{(k^2-u^2)}\sqrt{(1-u^2)}}$$

when $0 < u < k$, and

$$G(u \pm i0) = \frac{[-u^2\sqrt{(u^2-k^2)}\sqrt{(1-u^2)} \pm i(u^2-\frac{1}{2})^2]\sqrt{(1-u^2)}u}{(u^2-\frac{1}{2})^4 + u^4(u^2-k^2)(1-u^2)}$$

when $k < u < 1$.

If these are substituted into (2.6), (2.7), by (5.3), $R(r,t) = 0$ when $t <$

$(a^2 + r^2)^{1/2}/c_p$, and

$$
\begin{aligned}
R(r, t) = {} & \frac{V}{2\pi(1 - k^2)} \frac{r}{(a^2 + r^2)^{3/2}} \\
& + \frac{Vr}{4\pi a^3} \frac{\eta_0\sqrt{(\eta_0 - 1)}}{\Delta'(\eta_0)} \operatorname{Re} \{[\sqrt{(\eta_0 - k^2)} - i\tau]^2 + \eta_0 r^2/a^2\}^{-3/2} \\
& - \frac{Vr}{4\pi^2 a^3} \int_{k^2}^1 \operatorname{Re} \{[\sqrt{(\eta - k^2)} - i\tau]^2 + \eta r^2/a^2\}^{-3/2} \\
& \qquad \cdot \left\{ \frac{(\eta - \tfrac{1}{2})^2\sqrt{(1 - \eta)}\eta \, d\eta}{(\eta - \tfrac{1}{2})^4 + \eta^2(\eta - k^2)(1 - \eta)} \right\}
\end{aligned} \tag{5.7}
$$

when $t > (a^2 + r^2)^{1/2}/c_p$, where the restriction $c_s = 1$ has been removed by transforming to arbitrary units, and where

$$\tau = c_s t/a, \tag{5.8}$$

$$\Delta(\eta) = (\eta - \tfrac{1}{2})^2 - \eta\sqrt{(\eta - k^2)}\sqrt{(\eta - 1)}.$$

The reduction of (5.2) is similar. The final result is that $Z(r, t) = 0$ when $t < (a^2 + r^2)^{1/2}/c_p$, and

$$
\begin{aligned}
Z(r, t) = {} & \frac{V}{2\pi(1 - k^2)} \frac{a}{(a^2 + r^2)^{3/2}} \\
& + \frac{V}{4\pi a^3} \frac{\eta_0 - \tfrac{1}{2}}{\Delta'(\eta_0)} \operatorname{Re} \frac{\sqrt{(\eta_0 - k^2)} - i\tau}{\{[\sqrt{(\eta_0 - k^2)} - i\tau]^2 + \eta_0 r^2/a^2\}^{3/2}} \\
& - \frac{V}{4\pi^2 a^3} \int_{k^2}^1 \operatorname{Re} \frac{\sqrt{(\eta - k^2)} - i\tau}{\{[\sqrt{(\eta - k^2)} - i\tau]^2 + \eta r^2/a^2\}^{3/2}} \\
& \qquad \times \left\{ \frac{(\eta - \tfrac{1}{2})\sqrt{(\eta - k)}\sqrt{(1 - \eta)}\eta \, d\eta}{(\eta - \tfrac{1}{2})^4 + \eta^2(\eta - k^2)(1 - \eta)} \right\}.
\end{aligned} \tag{5.9}
$$

6. A proof similiar to that of §4, but not requiring condition iv) of §2 nor Lemmas 1 and 2 shows that

$$
\begin{aligned}
\lim_{\epsilon \to 0^+} \int_0^\infty e^{-\omega t} D \int_C J_0(\omega r s) \exp\left[-\omega \sum_1^n m p_m \lambda_m(s) \right] G(s) \, ds e^{i\omega t} \, d\omega \\
= DH(p_1, p_2, \cdots, p_n, r, t),
\end{aligned} \tag{6.1}
$$

where D signifies any product of the operators

$$1, \partial/\partial p_1, \partial/\partial p_2, \cdots, \partial/\partial p_n, \partial/\partial r, \partial/\partial t, \tag{6.2}$$

and where H is given in (2.6) and (2.7). The indicated differentiations under the integral signs may easily be justified when $\epsilon > 0$; (6.1) results on passing to the limit.

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THE DIRICHLET PROBLEM FOR MULTIPLY CONNECTED DOMAINS*

By R. R. Reynolds**

1. Introduction. The classical Dirichlet problem of determining a harmonic function from its values on the boundary of a given multiply connected domain occurs in various fields of applied mathematics. Bergman has developed a method for obtaining the solution of this problem as a series of orthogonal functions; however, although the mathematical theory is completely developed (see the bibliography in his book [1]), it is useful to modify his basic formulas (see (2.2) in the next section) in order to make them more convenient for evaluation on desk calculators, punch card machines, or other devices. This alteration is effected by expressing (2.2) as the real part of a complex function (see (2.6)) and making a number of linear substitutions culminating in formula (3.15).

In §2 pertinent results extracted from chapters 1 and 5 of [1] and §4 of [4] are summarized. In §3 formula (2.2) is transformed into (3.15). In §4 the steps that a computer must take in solving the problem are summarized. In §5 the integrals $D_{m,n}$ of (2.10b) are evaluated explicitly for the case where the domain is a circle with any number of circular holes punched out (see (5.4) and (5.8)). Section 6 contains the calculation of the 7-termed approximant to the harmonic function assuming the respective values 0, 1, and 0 on the circumferences of the exterior and left and right interior circles forming the boundaries of a given triply connected domain.

2. The method. Let (ξ, η) and (x, y) denote points in the domain $\mathfrak{B}$ or on its boundary $\mathfrak{b}$; $\mathfrak{B}$ is a plane, open, M-ply connected, bounded domain, while $\mathfrak{b}$ consists of M disjoint, smooth, simple, closed curves: $\mathfrak{b}_0$, and $\mathfrak{b}_1, \cdots \mathfrak{b}_{M-1}$, lying inside $\mathfrak{b}_0$. Also, let the symbol $[U, V]$ denote the integral

$$\iint_{\mathfrak{B}} \left\{ \frac{\partial U(x, y)}{\partial x} \frac{\partial V(x, y)}{\partial x} + \frac{\partial U(x, y)}{\partial y} \frac{\partial V(x, y)}{\partial y} \right\} dx\, dy.$$

Bergman has shown [1] that if $\{u(\xi, \eta)\}$ is the class of non-constant, single-valued, real, harmonic functions in $\mathfrak{B}$ which possess bounded integrals $[u, u]$, then,

i) there exists a sequence of linearly independent functions $\varphi_n(\xi, \eta)$ $(n = 1, 2, \cdots)$ belonging to the same class as u and satisfying the conditions of *completeness*

$$[u, u] - \sum_{n=1}^{2N+M-1} [u, \varphi_n]^2 \to 0 \quad \text{as} \quad N \to \infty$$

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and *orthogonality*

$$[\varphi_m, \varphi_n] = \delta_{m,n} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}, \qquad m, n = 1, 2, \cdots, \tag{2.1}$$

and

ii) $u(\xi, \eta)$ is approximated by

$$u^{(N)}(\xi, \eta) = \sum_{n=1}^{2N+M-1} \alpha_n \varphi_n(\xi, \eta), \qquad \alpha_n = [u, \varphi_n], \tag{2.2}$$

uniformly and absolutely in every closed subdomain of $\mathfrak{B}$. If also u is piecewise continuous on $\mathfrak{b}$, the coefficients α_n in (2.2) may be transformed by Green's theorem into line integrals

$$\alpha_n = \int_{\mathfrak{b}} u(x, y)[\partial\varphi_n(x, y)/\partial\nu]\, ds \tag{2.3}$$

which depend on the boundary values only of u. The integration proceeds counterclockwise along $\mathfrak{b}_0$ and in the opposite direction along each of the other $\mathfrak{b}_\mu$; $\partial\varphi_n/\partial\nu$ is the (outward) normal derivative

$$(\partial\varphi_n/\partial x)(dy/ds) - (\partial\varphi_n/\partial y)(dx/ds)$$

of φ_n.

Being harmonic, the $\varphi_n(\xi, \eta)$ in (2.2) are the real ($\Re$) and imaginary ($\Im$) parts of certain analytic functions $f_m(\zeta)$ of the complex variable $\zeta = \xi + i\eta$. Specifically, in a multiply connected domain ($M > 1$), the first $M - 1$ of the φ_n are taken to be

$$\varphi_\mu(\xi, \eta) = \Re f_\mu(\zeta), \qquad \mu = 1, \cdots M - 1, \tag{2.4a}$$

while the remaining $2N$ are paired as harmonic conjugates defined by

$$f_m(\zeta) = \varphi_{2m-M}(\xi, \eta) + i\varphi_{2m-M+1}(\xi, \eta), \qquad m = M, \cdots N + M - 1, \tag{2.4b}$$

whence it is seen that

i) the orthogonality condition (2.1) is equivalent to

$$\iint_{\mathfrak{B}} f_m'(z) f_n'^{*}(z)\, dx\, dy = \delta_{m,n} \qquad m, n = 1, 2, \cdots, \tag{2.5}$$

where the prime indicates the derivative, the star the complex conjugate,*** and $z = x + iy$; and

ii) formula (2.2) becomes

$$u^{(N)}(\xi, \eta) = \Re\left\{\sum_{\mu=1}^{M-1} (\Re a_\mu^*) f_\mu(\zeta) + \sum_{m=M}^{N+M-1} a_m^* f_m(\zeta)\right\}, \tag{2.6}$$

with the coefficients defined according to (2.2), (2.3), (2.4) by

$$\begin{aligned} \alpha_\mu &= \Re a_\mu, & \mu &= 1, \cdots M - 1, \\ a_m &= \alpha_{2m-M} + i\alpha_{2m-M+1}, & m &= M, \cdots, N + M - 1, \end{aligned} \tag{2.7a}$$

***For sake of shortness we write $f_n'^{*}(z)$ instead of $(f_n'(z))^*$.

or, since $f'(z) = \partial f(z)/\partial x = \partial f/\partial(iy)$, by

$$a_m = -i \int_b u(x, y) f_m'(z)\, dz, \qquad m = 1, 2, \cdots. \tag{2.7b}$$

The $f_n'(\zeta)$ are defined by

$$f_\mu'(\zeta) = \lim_{N\to\infty} f_\mu^{(N)\prime}(\zeta) \tag{2.8a}$$

$$f_\mu^{(N)\prime}(\zeta) = \sum_{\nu=1}^{\mu} C_{\mu,\nu}^{(N)} v_\nu'(\zeta) + \sum_{n=M}^{N+M-1} C_{\mu,n}^{(N)} v_n'(\zeta), \qquad \mu = 1, \cdots M-1 \tag{2.8b}$$

(see statement after (3.12) for proof of convergence),

$$f_m'(\zeta) = \sum_{n=M}^{m} C_{m,n} v_n'(\zeta), \qquad m = M, \cdots N+M-1, \tag{2.8c}$$

where the functions on the right are the derivatives of the linearly independent functions.

$$\begin{aligned} v_\nu(\zeta) &= \log_e(\zeta - \zeta_\nu) && \nu = 1, \cdots M-1 \\ v_{pM}(\zeta) &= \zeta^p/p \\ v_{pM+\nu}(\zeta) &= -(\zeta - \zeta_\nu)^{-p}/p && p = 1, 2, \cdots \end{aligned} \tag{2.9}$$

(ζ_ν is an arbitrary fixed point in the open domain (outside $\mathfrak{B}$) bounded by b_ν), and the coefficients are determined uniquely by substituting (2.8) into (2.5). This orthogonalization process will be carried out in the next section where it will be shown that $u^{(N)}(\xi, \eta)$ is the real part of a linear function of the $v_n(\zeta)$ with coefficients depending on the integrals

$$\text{a)}\quad c_n = -i \int_b u(x, y) v_n'(z)\, dz, \qquad \text{b)}\quad D_{m,n} = \iint_{\mathfrak{B}} v_m'(z) v_n'^*(z)\, dx\, dy. \tag{2.10}$$

3. Orthogonalization. The necessary substituting will be facilitated by using matrices. Let column and row matrices be represented by small boldface letters; e. g.,

$$\mathbf{f}_0(\zeta) = \begin{pmatrix} f_1(\zeta) \\ \cdot \\ \cdot \\ \cdot \\ f_{M-1}(\zeta) \end{pmatrix}, \quad \mathbf{f}(\zeta) = \begin{pmatrix} f_M(\zeta) \\ \cdot \\ \cdot \\ \cdot \\ f_{N+M-1}(\zeta) \end{pmatrix}, \qquad \begin{aligned} \mathbf{f}_0^*(\zeta) &= (f_1^*(\zeta), \cdots f_{M-1}^*(\zeta)), \\ \mathbf{f}^*(\zeta) &= (f_M^*(\zeta), \cdots f_{N+M-1}^*(\zeta)), \end{aligned}$$

and rectangular (and square) ones by capital boldface letters; e.g.,

$$\mathbf{D}_{00} = \begin{pmatrix} D_{1,1} & & \\ & \ddots & \\ & & D_{M-1,M-1} \end{pmatrix}, \qquad \mathbf{D}_0^* = \begin{pmatrix} D_{1,M} & & \\ & \ddots & \\ & & D_{M-1,N+M-1} \end{pmatrix},$$

$$\mathbf{D}_0 = \begin{pmatrix} D_{M,1} & & \\ & \ddots & \\ & & D_{N+M-1,M-1} \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} D_{M,M} & & \\ & \ddots & \\ & & D_{N+M-1,N+M-1} \end{pmatrix}.$$

The presence or absence of zero subscripts appended to a matrix indicates the range of the subscripts of its elements; a star designates the transposed matrix with its elements replaced by their complex conjugates. The definition of $\mathbf{D}_0^*$ above is consistent with this convention since all the integrals (2.10b) satisfy the hermitian relation $D_{m,n} = D_{n,m}^*$.

Now let

$$\mathbf{f}'(\zeta), \qquad \mathbf{a} = -i\int_b u(x, y)\mathbf{f}'(z)\,dz, \qquad \mathbf{D} = \iint_{\mathfrak{B}} \mathbf{v}'(z)\mathbf{v}'^*(z)\,dx\,dy, \qquad \Re\mathbf{a},$$

etc. denote the matrices with respective general elements

$$f_m'(\zeta), \qquad a_m = -i\int_b u(x, y)f_m'(z)\,dz,$$

$$D_{m,n} = \iint_{\mathfrak{B}} v_m'(z)v_n'^*(z)\,dx\,dy, \qquad \Re a_m, \qquad m = M, \cdots N + M - 1,$$

etc; then (2.5), (2.6), and (2.8) become, in matrix form,

$$\text{a)} \quad \iint_{\mathfrak{B}} \mathbf{f}_0^{(N)\prime}\mathbf{f}_0^{(N)\prime *}\,dx\,dy = \mathbf{I}, \qquad \text{b)} \quad \iint_{\mathfrak{B}} \mathbf{f}_0^{(N)\prime}\mathbf{f}'^*\,dx\,dy = \mathbf{O},$$

(3.1)

$$\iint_{\mathfrak{B}} \mathbf{f}'\mathbf{f}_0^{(N)\prime *}\,dx\,dy = \mathbf{O}, \qquad \text{c)} \quad \iint_{\mathfrak{B}} \mathbf{f}'\mathbf{f}'^*\,dx\,dy = \mathbf{I};$$

$$u^{(N)}(\xi, \eta) = \Re\{(\Re\,\mathbf{a}_0^{(N)*})\mathbf{f}_0^{(N)}(\zeta) + \mathbf{a}^*\mathbf{f}(\zeta)\}; \tag{3.2}$$

and

$$\mathbf{f}_0^{(N)\prime}(\zeta) = \mathbf{C}_{00}^{(N)}\mathbf{v}_0'(\zeta) + \mathbf{C}_0^{(N)*}\mathbf{v}'(\zeta), \tag{3.3a}$$

$$\mathbf{f}'(\zeta) = \mathbf{C}\mathbf{v}'(\zeta), \tag{3.3b}$$

respectively, with $\mathbf{I}$ and $\mathbf{O}$ appropriate unit and zero matrices, and

$$\text{a)}\ \mathbf{a}_0^{(N)} = \mathbf{C}_{00}^{(N)}\mathbf{c}_0 + \mathbf{C}_0^{(N)*}\mathbf{c}, \qquad \text{b)}\ \mathbf{a} = \mathbf{C}\mathbf{c}. \tag{3.4}$$

Substitution of (3.3b) in (3.1c) yields

$$\iint_{\mathfrak{B}} \mathbf{C}\mathbf{v}'\mathbf{v}'^*\mathbf{C}^*\,dx\,dy = \mathbf{C}\iint_{\mathfrak{B}} \mathbf{v}'\mathbf{v}'^*\,dx\,dy\mathbf{C}^* = \mathbf{C}\mathbf{D}\mathbf{C}^* = \mathbf{I}. \tag{3.5}$$

It follows from the linear independence of the $v_n'(\zeta)$ that none of the diagonal elements of $\mathbf{C}$ can be zero; therefore, since $\mathbf{C}$ is triangular, having zero elements above its diagonal, its inverse $\bar{\mathbf{C}}$ exists and

$$\bar{\mathbf{C}}\bar{\mathbf{C}}^* = \mathbf{D}, \qquad \bar{\mathbf{C}}\mathbf{C} = \mathbf{I}. \tag{3.6}$$

From these are derived scalar equations for the successive determination of the $C_{m,n}$:

$$\bar{C}_{M,M} = (D_{M,M})^{\frac{1}{2}}, \qquad \bar{C}_{n,n} = \left(D_{n,n} - \sum_{k=M}^{n-1} |\bar{C}_{n,k}|^2\right)^{\frac{1}{2}},$$

$$\bar{C}_{m,M} = D_{m,M}/\bar{C}_{M,M}, \qquad \bar{C}_{m,n} = \left(D_{m,n} - \sum_{k=M}^{n-1} \bar{C}_{m,k}\bar{C}_{n,k}^*\right)\Big/\bar{C}_{n,n} \tag{3.7a}$$

with (m, n) taken in the order $(M, M), \cdots (N + M - 1, M); (M + 1, M + 1), \cdots (N + M - 1, M + 1); \cdots (N + M - 1, N + M - 1)$, and

$$C_{m,m} = 1/\tilde{C}_{m,m}, \qquad C_{m,n} = -C_{m,m}\sum_{k=n}^{m-1} \tilde{C}_{m,k}C_{k,n} \tag{3.7b}$$

with $(m, n) = (M, M); (M + 1, M + 1), (M + 1, M); (M + 2, M + 2), \cdots (M + 2, M); \cdots (N + M - 1, N + M - 1), \cdots (N + M - 1, M)$.

According to (3.8) below, $\tilde{\mathbf{D}}^{(N)} = \mathbf{C}^*\mathbf{C}$, the inverse of $\mathbf{D}$, is a positive definite matrix; therefore, the radicands above are positive.

The formulas (3.7) incidentally are essentially those used by Banachiewicz in his "square root" method of inverting $\mathbf{D}$, and are convenient for computation on desk calculating machines; this method, employing auxiliary matrices and check columns, is described in [5]. It will be seen subsequently that $u^{(N)}$ does not really depend on $\mathbf{C}$ but rather on $\tilde{\mathbf{D}}^{(N)}$; consequently, other methods of inversion, including those adaptable to punch card machines, may be used. The elements of $\tilde{\mathbf{D}}^{(N)}$ vary with N; however, their convergence is assured from the fact that

$$K^{(N)}(\zeta, z^*) = \mathbf{f}'^*(z)\mathbf{f}'(\zeta) = \mathbf{v}'^*(z)\mathbf{C}^*\mathbf{C}\mathbf{v}'(\zeta) = \mathbf{v}'^*(z)\mathbf{D}^{(N)}\mathbf{v}'(\zeta) \tag{3.8}$$

converges uniformly and absolutely in every closed subdomain of B to an analytic function—Bergman's kernel function $K(\zeta, z^*)$—of two complex variables.

Substitution of (3.3) in (3.1b) and (3.1a) yields

$$\mathbf{C}_0^{(N)*} = -\mathbf{C}_{00}^{(N)}\mathbf{D}_0^*\tilde{\mathbf{D}}^{(N)} \tag{3.9}$$

and

$$\mathbf{C}_{00}^{(N)}\,\mathbf{E}_{00}^{(N)}\,\mathbf{C}_{00}^{(N)*} = \mathbf{I}, \tag{3.10}$$

where

$$\mathbf{E}_{00}^{(N)} = \mathbf{D}_{00} - \mathbf{D}_0^*\tilde{\mathbf{D}}^{(N)}\mathbf{D}_0. \tag{3.11}$$

Because of the formal analogy between (3.10) and (3.5), the $C_{\mu,\nu}^{(N)}$ are obtained from

$$\tilde{C}_{1,1}^{(N)} = (E_{1,1}^{(N)})^{\frac{1}{2}}, \qquad \tilde{C}_{\nu,\nu}^{(N)} = \left(E_{\nu,\nu}^{(N)} - \sum_{k=1}^{\nu-1} |\tilde{C}_{\nu,k}^{(N)}|^2\right)^{\frac{1}{2}},$$

$$\tilde{C}_{\mu,1}^{(N)} = E_{1,\mu}^{(N)}/\tilde{C}_{1,1}^{(N)}, \qquad \tilde{C}_{\mu,\nu}^{(N)} = \left(E_{\mu,\nu}^{(N)} - \sum_{k=1}^{\nu-1} \tilde{C}_{\mu,k}^{(N)}\tilde{C}_{\nu,k}^{(N)*}\right)\Big/\tilde{C}_{\nu,\nu}^{(N)} \tag{3.12a}$$

with $(\mu, \nu) = (1, 1), \cdots (M - 1, 1); (2, 2), \cdots (M - 1, 2); \cdots (M - 1, M - 1)$, and

$$C_{\mu,\mu}^{(N)} = 1/\tilde{C}_{\mu,\mu}^{(N)}, \qquad C_{\mu,\nu}^{(N)} = -C_{\mu,\mu}^{(N)}\sum_{k=\nu}^{\mu-1} \tilde{C}_{\mu,k}^{(N)}\,C_{k,\nu}^{(N)}, \tag{3.12b}$$

with $(\mu, \nu) = (1, 1); (2, 2), (2, 1); (3, 3), \cdots (3, 1); \cdots (M - 1, M - 1), \cdots$

$(M-1, 1)$; the convergence of the $C_{\mu,\nu}^{(N)}$ follows from that of the $E_{\mu,\nu}^{(N)}$ of (3.11). The second term of (3.3a) is

$$-\mathbf{C}_{00}^{(N)}\mathbf{D}_0^*\tilde{\mathbf{D}}^{(N)}\mathbf{v}'(\zeta) = -\mathbf{C}_{00}^{(N)} \iint_{\mathfrak{B}} \mathbf{v}_0'(z)\mathbf{v}'^*(z)\tilde{\mathbf{D}}^{(N)}\mathbf{v}'(\zeta)\, dx\, dy$$

$$= -C_{00}^{(N)} \iint_{\mathfrak{B}} \mathbf{v}_0'(z)K^{(N)}(\zeta, z^*)\, dx\, dy;$$

hence the $f_\mu^{(N)\prime}$ of (2.8b) converge, when $N \to \infty$.

Substitution of (3.3) and (3.4) in (3.2), with the help of (3.9), yields the desired formula

$$u^{(N)}(\xi, \eta) = \alpha^{(N)} + \tilde{u}^{(N)}(\xi, \eta), \tag{3.15a}$$

$$\tilde{u}^{(N)}(\xi, \eta) = \Re\{\Re\{(\mathbf{c}_0^* - \mathbf{c}^*\tilde{\mathbf{D}}^{(N)}\mathbf{D}_0)\mathbf{C}_{00}^{(N)*}\}\mathbf{C}_{00}^{(N)} \times [\mathbf{v}_0(\zeta) - \mathbf{D}_0^*\tilde{\mathbf{D}}^{(N)}\mathbf{v}(\zeta)] + \mathbf{c}^*\tilde{\mathbf{D}}^{(N)}\mathbf{v}(\zeta)\}, \tag{3.15b}$$

where $\alpha^{(N)}$ is a constant of integration.

4. Summary. After choosing N, compute $D_{m,n}$ (2.10b), where the v_m are defined in (2.9), and arrange them in the matrices $\mathbf{D}$, $\mathbf{D}_0$, $\mathbf{D}_{00}$. Compute the inverse $\tilde{\mathbf{D}}^{(N)} = \mathbf{C}^*\mathbf{C}$ by using the formulas (3.7) or some other method. Compute $\mathbf{E}_{00}^{(N)}$ (3.11) and $\mathbf{C}_{00}^{(N)}$ (3.12); if the elements of $\mathbf{C}_{00}^{(N)}$ are real, then the product $\mathbf{C}_{00}^{(N)*}\mathbf{C}_{00}^{(N)} = \tilde{\mathbf{E}}_{00}^{(N)}$ appears and it is necessary to calculate only the inverse of $\mathbf{E}_{00}^{(N)}$.

Compute the line integrals (2.10a) and arrange in the matrices $\mathbf{c}_0$ and $\mathbf{c}$. Compute $\Re\{(\mathbf{c}_0^* - \mathbf{c}^*\tilde{\mathbf{D}}^{(N)}\mathbf{D}_0)\mathbf{C}_{00}^{(N)*}\}$ and $\mathbf{c}^*\tilde{\mathbf{D}}^{(N)}$. Compute the functions v_m along $\mathfrak{b}$ and arrange in the matrices $\mathbf{v}_0$ and $\mathbf{v}$.

Compute $\tilde{u}^{(N)}(x, y)$ and $d^{(N)}(x, y) \equiv u(x, y) - \tilde{u}^{(N)}(x, y)$ at a number of points (x, y) on $\mathfrak{b}$. Since $\tilde{u}^{(N)}(\xi, \eta)$ is a linear combination of harmonic functions at all points (ξ, η) of $\mathfrak{B}$, $d^{(N)}(\xi, \eta)$ is also harmonic and consequently assumes its extrema $d_{\max}^{(N)}$ and $d_{\min}^{(N)}$ on $\mathfrak{b}$; thus, for any choice of $\alpha^{(N)}$, the limits of error are known. In particular, if this constant is defined by $\alpha^{(N)} = (d_{\max}^{(N)} + d_{\min}^{(N)})/2$, then $|\max\{u(x, y) - u^{(N)}(x, y)\}| = |\min\{u(x, y) - u^{(N)}(x, y)\}|$. On the other hand, if $\alpha^{(N)}$ is defined in a similar way in terms of the extreme values of $d^{(N)}(x, y)$ on a subset $\bar{\mathfrak{b}}$ of $\mathfrak{b}$, then the deviation is larger along part of $\mathfrak{b}$, but may be rather small on $\bar{\mathfrak{b}}$. It is possible to reduce the error by increasing N, but this requires computation from the beginning. A more efficient method is now described. The function $d^{(N)}(x, y)$ has a number of relative maxima and minima $d_m^{(N)}$ $(m = 1, \cdots r$, say) at the points (x_m, y_m). For each m construct a harmonic function $u_m^{(N)}(x, y)$ which assumes the value $-d_m^{(N)}$ at (x_m, y_m) and approaches zero at other points; as an example

$$c \log_e\{(x - x_m + \epsilon_m^{(1)})^2 + (y - y_m + \epsilon_m^{(2)})^2\}$$

is such a function for an appropriate choice of the constant c and the small constants $\epsilon_m^{(1)}$, $\epsilon_m^{(2)}$, which are selected so that the points $(x_m - \epsilon_m^{(1)}, y_m - \epsilon_m^{(2)})$

lie just outside $\mathfrak{B}$. Generally then $u^{(N)}(\xi, \eta) + \sum_{m=1}^{r} u_m^{(N)}(\xi, \eta)$ is a closer approximation than $u^{(N)}(\xi, \eta)$ alone. The procedure of constructing functions $u_m^{(N)}(x, y)$ for a simply connected domain has been discussed in [2] and [3].

5. A domain with circular boundaries. In general, the integrals in (2.10b) may be evaluated by the means of identity

$$\iint_{\mathfrak{B}} v_r'(z)v_s'(z)\,dx\,dy = (2i)^{-1}\int_{\mathfrak{b}} v_r'(z)v_s^*(z)\,dz, \tag{5.1}$$

valid for functions $v_r'v_s^*$ single-valued in $\mathfrak{B}$ (z^* is a function of z on $\mathfrak{b}$). Either side of (5.1) may always be integrated numerically; however, if $\mathfrak{B}$ is the domain consisting of the circle interior to $\mathfrak{b}_0$ ($|z| = R_0$) with the $M-1$ circles $\mathfrak{b}_\lambda$ ($|z - \zeta_\lambda| = R_\lambda$) punched out, then

$$z^* = R_0^2/z \text{ on } \mathfrak{b}_0\,, \quad z^* = \zeta_\lambda^* + R_\lambda^2/(z - \zeta_\lambda) \text{ on } \mathfrak{b}_\lambda \quad (\lambda = 1, 2, \cdots M-1), \tag{5.2}$$

and the right-hand side of (5.1) reduces to a form which may be integrated with the help of Cauchy's integral formulas

$$\frac{1}{2i}\int_{\mathfrak{b}} \frac{h(z)}{(z-\zeta)^{k+1}}\,dz = \frac{\pi}{k!}\left[\frac{d^k h(z)}{dz^k}\right]_{z=\zeta}, \qquad k = 0, 1, \cdots \tag{5.3}$$

(ζ inside $\mathfrak{b}$) valid for $h(z)$ analytic inside $\mathfrak{b}$. Consequently, the elements of $\mathbf{D}$ and $\mathbf{D}_0$ ($p = 0, 1, \cdots$; $q = 1, 2 \cdots$) are given by

$$D_{pM+\mu,qM+\nu} = (\pi/q)W_{\mu,\nu}^{(p,q)} - \sum_{\lambda=0}^{M-1}{}' P_{\lambda,\mu,\nu}^{(p,q)}\,, \qquad \mu, \nu = 0, 1, \cdots M-1 \tag{5.4}$$

where $\sum'$ indicates that $\lambda = \mu$ and $\lambda = \nu$ are omitted from the summation, and

$$P_{\lambda,\mu,\nu}^{(p,q)} = Z_{\lambda,\mu,\nu}^{(p,q)} \sum_{k=0}^{k(\lambda)} \beta_{\lambda,\mu,\nu}^{(k,p,q)}(Y_{\lambda,\mu,\nu})^k, \qquad k(\lambda) = \begin{cases} q-1 & \text{if } \lambda \neq 0 \\ p & \text{if } \lambda \neq 0. \end{cases} \tag{5.5}$$

The quantities appearing are displayed in Tables 1 and 2 (the sign $=$ or $\neq$ indicates whether λ, μ, or ν is zero or not; some of the $P_{\lambda,\mu,\nu}^{(p,q)}$ are modified if certain $\zeta_\lambda = 0$; see (5.7)).

The computation may be arranged as follows: Let λ, μ, ν be chosen from the numbers 0, 1, 2, $\cdots M-1$ with the restrictions $\lambda \neq \mu$, $\lambda \neq \nu$. Arrange the allowable ordered triples (λ, μ, ν) into the groups

$$\text{I } (\lambda, 0, 0), \quad \text{II } (\lambda, \mu, 0), \quad \text{III } (\lambda, \mu, \nu), \quad \text{III}_0(0, \mu, \nu); \quad \lambda, \mu, \nu \neq 0. \tag{5.6}$$

Number a set of long sheets according to this scheme and arrange them according to groups. Rule eight columns on each sheet and enter the quantities p, q, k, $\beta_{\lambda,\mu,\nu}^{(k,p,q)}$, $(Y_{\lambda,\mu,\nu})^k$, $\sum_{k=0}^{k(\lambda)} \beta_{\lambda,\mu,\nu}^{(k,p,q)}(Y_{\lambda,\mu,\nu})^k$, $Z_{\lambda,\mu,\nu}^{(p,q)}$, $P_{\lambda,\mu,\nu}^{(p,q)}$ (take p, q, k in the order suggested in Table 2). Number another set of sheets according to the pairs (μ, ν) and enter the quantities $\sum_{\lambda=0}^{\prime M-1} P_{\lambda,\mu,\nu}^{(p,q)}$, $W_{\mu,\nu}^{(p,q)}$,

$$W_{\mu,\nu}^{(p,q)} - \sum_{\lambda=0}^{\prime M-1} P_{\lambda,\mu,\nu}^{(p,q)}\,, D_{pM+\mu,qM+\nu} \qquad (\mu, \nu = 0, 1, \cdots M-1).$$

TABLE 1

	λ	μ	ν	$\beta^{(k,p,q)}_{\lambda,\mu,\nu}$	$Y_{\lambda,\mu,\nu}$	$Z^{(p,q)}_{\lambda,\mu,\nu}$	$W^{(p,q)}_{\mu,\nu}$
I	$\neq$	$=$	$=$	$\binom{p-1}{k}\binom{q}{k+1}$	$\left\lvert\frac{R_\lambda}{\zeta_\lambda}\right\rvert^2$	$R_\lambda^2\zeta_\lambda^{p-1}\zeta_\lambda^{*q-1}$	$\delta_{q,p}R_0^{2q}$
II	$\neq$	$\neq$	$=$	$\binom{p+k}{p}\binom{q}{k+1}$	$\frac{R_\lambda^2/\zeta_\lambda^*}{\zeta_\mu-\zeta_\lambda}$	$\frac{R_\lambda^2\zeta_\lambda^{*q-1}}{(\zeta_\lambda-\zeta_\mu)^{p+1}}$	$-\delta_{p,0}\zeta_\mu^{*q}$
III	$\neq$	$\neq$	$\neq$	$\binom{p+k}{p}\binom{q}{k+1}$	$\frac{R_\lambda^2}{\zeta_\lambda-\zeta_\mu)(\zeta_\lambda^*-\zeta_\nu^*)-R_\lambda^2}$	$\frac{R_\lambda^2(\zeta_\lambda^*-\zeta_\nu^*)^{p-q}}{[(\zeta_\lambda-\zeta_\mu)(\zeta_\lambda^*-\zeta_\nu^*)-R_\mu^2]^{p+1}}$	$\delta_{p,q}/R_\mu^{2q}$, $\mu=\nu$
III.	$=$	$\neq$	$\neq$	$\binom{q+k-1}{q-1}\binom{q}{p-k}$	$\frac{\zeta_\mu\zeta_\nu^*}{R_0^2-\zeta_\mu\zeta_\nu^*}$	$\frac{\zeta_\mu^{q-p}}{(R_0^2-\zeta_\mu\zeta_\nu^*)^q}$	$\delta_{p,0}/(\zeta_\mu^*-\zeta_\nu^*)^q$, $\mu\neq\nu$

TABLE 2

(I) $\beta_{\lambda,0,0}^{(k,p,q)}$, $\lambda \neq 0$ (II, III) $\beta_{\lambda,\mu,\nu}^{(\kappa,p,q)}$, $\lambda, \mu \neq 0$ (III$_2$) $\beta_{0,\mu,\nu}^{(k,p,q)}$, $\mu, \nu \neq 0$

		p											
q	k	1	2	3	4	5	0	1	2	3	4	5
1	0	1	1	1	1	1	1	1	1	1	1	1
2	0	2	2	2	2	2	2	2	2	2	2	2
2	1	0	1	2	3	4	1	2	3	4	5	6
3	0	3	3	3	3	3	3	3	3	3	3	3
3	1	0	3	6	9	12	3	6	9	12	15	18
3	2	0	0	1	3	6	1	3	6	10	15	21
4	0	4	4	4	4	4	4	4	4	4	4	4
4	1	0	6	12	18	24	6	12	18	24	30	36
4	2	0	0	4	12	24	4	12	24	40	60	84
4	3	0	0	0	1	4	1	4	10	20	35	56
5	0	5	5	5	5	5	5	5	5	5	5	5
5	1	0	10	20	30	40	10	20	30	40	50	60
5	2	0	0	10	30	60	10	30	60	100	150	210
5	3	0	0	0	5	20	5	20	50	100	175	280
5	4	0	0	0	0	1	1	5	15	35	70	126

		q				
p	k	1	2	3	4	5
0	0	1	1	1	1	1
1	0	1	2	3	4	5
1	1	1	2	3	4	5
2	0	0	1	3	6	10
2	1	1	4	9	16	25
2	2	1	3	6	10	15
3	0	0	0	1	4	10
3	1	0	2	9	24	50
3	2	1	6	18	40	75
3	3	1	4	10	20	35
4	0	0	0	0	1	5
4	1	0	0	3	16	50
4	2	0	3	18	60	150
4	3	1	8	30	80	175
4	4	1	5	15	35	70
5	0	0	0	0	0	1
5	1	0	0	0	4	25
5	2	0	0	6	40	150
5	3	0	4	30	120	350
5	4	1	10	45	140	350
5	5	1	6	21	56	126

In case some $\zeta_\lambda = 0$, replace the corresponding $P^{(p,q)}_{\lambda,\mu,\nu}$ by

$$\text{(5.7a)} \qquad \delta_{p,q} R_\lambda^{2q} \cdot \text{ or } \binom{p+q-1}{p} (-)^{p+1} R_\lambda^{2q}/\zeta_\mu^{p+q} \qquad \delta_{p,q} = \begin{cases} 0, & p \neq q \\ 1, & p = q \end{cases}$$

depending on whether (λ, μ, ν) is in group I or II, respectively; otherwise the formula in Table 1 is applicable. In case $\zeta_\mu = 0$, replace $P^{(p,q)}_{0,\mu,\nu}$ in group III_0 by

$$\text{(5.7b)} \qquad \binom{p-1}{q-1} \zeta_\nu^{p+q}/R_0^{2p} \text{ if } \mu \neq \nu, \qquad \text{or } \delta_{p,q}/R_0^2 \text{ if } \mu = \nu.$$

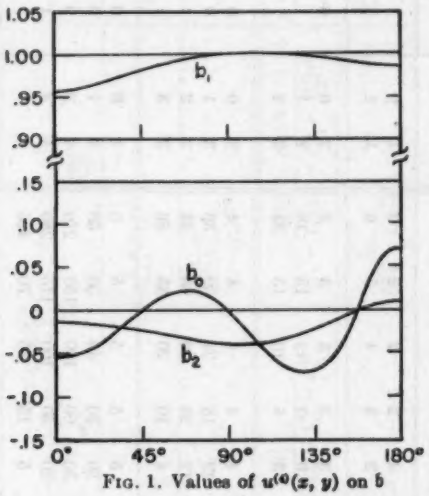

FIG. 1. Values of $u^{(0)}(x, y)$ on b

In D_{00} each v_λ $(\lambda = 1, \cdots M-1)$ is multivalued and (5.1) must be applied to the $(M-1)$-ply connected region formed by making a radial cut from b_λ to b_0. A little calculation yields

$$\text{(5.8)} \qquad \begin{aligned} D_{\mu,\mu} &= \pi \log_e\left(\frac{R_0^2 - |\zeta_\mu|^2}{R_\mu^2}\right) - \pi \sum_{\lambda=1}^{M-1}{}' \frac{R_\lambda^2}{|\zeta_\lambda - \zeta_\mu|^2}, \\ D_{\mu,\nu} &= \pi \log_e\left(\frac{R_0^2 - \zeta_\mu \zeta_\nu^*}{|\zeta_\mu - \zeta_\nu|^2}\right) - \pi \sum_{\lambda=1}^{M-1}{}' \frac{R_\lambda^2}{(\zeta_\lambda - \zeta_\mu)(\zeta_\lambda^* - \zeta_\nu^*)}, \qquad \mu \neq \nu. \end{aligned}$$

6. A numerical example. Consider the triply connected domain whose outer boundary is the circle b_0 with radius 4 and center at the origin and whose inner boundaries are the circles b_1 and b_2 with unit radii and centers at $\zeta = \mp 2$. First the integrals $D_{m,n}$ are calculated with the data

$$M = 3, \quad N = 4, \quad R_0 = 4, \quad R_1 = R_2 = 1, \quad \zeta_1 = -2, \quad \zeta_2 = 2.$$

After computing the appropriate matrices in (3.15), the following 7-termed approximant to the harmonic measure $u(\xi, \eta)$, which $=0$ on $\mathfrak{b}_0$ and $\mathfrak{b}_2$, and $=1$ on $\mathfrak{b}_1$, is obtained:

$$u^{(4)}(\xi, \eta) = 1.1160 + \Re\{-.97609 \log_e(\zeta + 2) + .15932 \log_e(\zeta - 2) + .14202\zeta - .14106(\zeta + 2)^{-1} + .12004(\zeta - 2)^{-1} - .00835\zeta^2\},$$

where $\alpha^{(4)} = \frac{1}{2}(d^{(4)}_{\max} + d^{(4)}_{\min})$. Fig. 1 and Table 3 show the values of $u^{(4)}(x, y)$ along different segments of $\mathfrak{b}$, the angles being measured counterclockwise from the right end of the horizontal diameter of each circle $\mathfrak{b}_\mu (\mu = 0, 1, 2)$. The

TABLE 3

° ′	$\mathfrak{b}_0$	$\mathfrak{b}_1$	$\mathfrak{b}_2$	° ′
00 00	−.0515	.9596	−.0122	360 00
11 15	−.0480	.9616	−.0131	348 45
22 30	−.0383	.9662	−.0154	337 30
33 45	−.0190	.9733	−.0192	326 15
45 00	−.0013	.9815	−.0239	315 00
56 15	+.0148	.9880	−.0292	303 45
67 30	+.0219	.9963	−.0343	292 30
78 45	+.0169	1.0014	−.0385	281 15
90 00	+.0001	1.0044	−.0409	270 00
101 15	−.0350	1.0051	−.0408	258 45
112 30	−.0517	1.0040	−.0374	247 30
123 45	−.0704	1.0013	−.0308	236 15
135 00	−.0714	.9959	−.0213	225 00
146 15	−.0475	.9943	−.0103	213 45
157 30	−.0007	.9900	+.0000	202 30
168 45	+.0505	.9890	+.0075	191 15
180 00	+.0737	.9882	+.0103	180 00

error does not exceed 7.4% and may be diminished by adding more terms as explained in §4.

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ON THE SOLUTION OF FREDHOLM INTEGRAL EQUATIONS OF SECOND KIND BY ITERATION

By Carl Wagner

1. Introduction. This paper deals with the numerical solution of Fredholm integral equations of second type,

$$\varphi(x) = f(x) - \lambda \int_a^b K(x, z)\varphi(z)\, dz \tag{1}$$

where a, b, and λ are given constant values, $f(x)$ and $K(x, z)$ are given functions, and $\varphi(x)$ is to be determined.

The solution $\varphi(x)$ may be obtained by an iteration process. We assume that an approximate solution $\psi^{(0)}(x)$ is known and we ask how a closer approximation can be obtained. Then the final solution may be written as the limiting value of successive approximations

$$\psi^{(n+1)}(x) = \psi^{(n)}(x) + k^{(n+1)}(x) \tag{2}$$

where $k^{(n+1)}(x)$ represents a correction term to be added to the preceding approximation $\psi^{(n)}(x)$. Hence

$$\varphi(x) = \psi^{(0)}(x) + k^{(1)}(x) + k^{(2)}(x) + \cdots . \tag{3}$$

Each correction term may be set proportional to

$$\Delta^{(n)}(x) = \psi^{(n)}(x) + \lambda \int K(x, z)\psi^{(n)}(z)\, dz - f(x), \tag{4}$$

i.e., the difference of the left-hand and the right-hand member of Eq. (1) if $\varphi(x)$ is replaced by the preceding approximation $\psi^{(n)}(x)$. Thus

$$k^{(n+1)}(x) = c(x)\Delta^{(n)}(x) = c(x)\,[\psi^{(n)}(x) + \lambda \int K(x, z)\psi^{(n)}(z)\, dz - f(x)] \tag{5}$$

where $c(x)$ is a factor to be determined below. Here and in the following equations the limits of integration, a and b, are omitted for the sake of brevity.

Setting $\psi^{(0)}(x) = f(x)$ and $c(x) = -1$, we obtain the Liouville-Neumann series. This series is, however, expedient for computations only if for a given kernel $K(x, z)$ the value of λ is sufficiently small.

To improve convergence, Wiarda[1] and Bückner[2] have suggested a modified sequence of approximations by using a factor $-c(x) = \Theta < 1$, independent of x. This follows readily from Eq. (3′) of Bückner's paper. Use of a factor $c(x)$ depending on x may therefore be considered as a generalization of Bückner's method.

2. Determination of the Factor $c(x)$. Tentatively, we may set

$$c(x) = -1/[1 + \lambda \int K(x, z)\, dz]. \tag{6}$$

Upon substitution of Eqs. (5) and (6) in Eq. (2), it follows that

$$\psi^{(n+1)}(x) = \frac{f(x) + \psi^{(n)}(x)\lambda \int K(x, z)\, dz - \lambda \int K(x, z)\psi^{(n)}(z)\, dz}{1 + \lambda \int K(x, z)\, dz}. \tag{7}$$

The special suggestion for the factor $c(x)$ in Eq. (6) may be justified by the following considerations.

Assuming a constant value for $\psi^{(0)}(x)$, we obtain from Eq. (7) as first iteration

$$\psi^{(1)}(x) = f(x) \Big/ \left[1 + \lambda \int K(x, z)\, dz\right], \tag{8}$$

regardless of the special constant value substituted for $\psi^{(0)}(x)$.

If the kernel $K(x, z)$ vanishes except for $x = z$, integral Eq. (1) becomes a linear algebraic equation in $\varphi(x)$,

$$\varphi(x) = f(x) - \varphi(x)\lambda \int K(x, z)\, dz \tag{9}$$

with the obvious solution

$$\varphi(x) = f(x) \Big/ \left[1 + \lambda \int K(x, z)\, dz\right] \tag{10}$$

which is equal to the first iteration obtained in Eq. (8). Hence we conclude that successive approximations calculated with the aid of Eq. (7) will converge rather rapidly if the kernel $K(x, z)$ has a pronounced maximum for $x = z$ so that the behavior of integral Eq. (1) is similar to that of algebraic Eq. (9).

It is obvious that Eq. (6) does not apply if $\lambda \int K(x, z)\, dz = -1$ for any value of x. By and large, use of Eq. (6) does not seem recommendable if $\lambda \int K(x, z)\, dz$ lies between -1.5 and -0.5, but the practical limits depend on the characteristics of $K(x, z)$ for $x \sim z$. So far Eq. (6) has been applied only to integral equations in which $\lambda \int K(x, z)\, dz$ is positive.

3. Derivation of a Criterion of Convergence. We denote by $\epsilon^{(n)}(x)$ the deviation of the n^{th} approximation from the true solution,

$$\epsilon^{(n)}(x) = \psi^{(n)}(x) - \varphi(x). \tag{11}$$

Upon substitution of Eq. (11) in Eq. (5), it follows that

$$k^{(n+1)}(x) = c(x)\left[\varphi(x) + \epsilon^{(n)}(x) + \lambda \int K(x, z)\varphi(z)\, dz + \lambda \int K(x, z)\epsilon^{(n)}(z)\, dz - f(x)\right], \tag{12}$$

whence upon comparison to Eq. (1),

$$k^{(n+1)}(x) = c(x)\left[\epsilon^{(n)}(x) + \lambda \int K(x, z)\epsilon^{(n)}(z)\, dz\right]. \tag{13}$$

If $\epsilon^{(n)}(x)$ is independent of x, i.e., the approximate solution $\psi^{(n)}(x)$ deviates from the exact solution by a constant value, then $k^{(n+1)}(x)$ equals $-\epsilon^{(n)}$, and $\psi^{(n+1)}(x)$ becomes $\varphi(x)$, i.e., the exact solution is reached in one step. A constant value of $\epsilon^{(n)}(x)$ is, of course, not to be expected in practical computations, but $k^{(n+1)}(x)$ will be rather close to $-\epsilon^{(n)}(x)$ if $\epsilon^{(n)}(x)$ is a smooth function of x and $K(x, z)$ has a high peak for $x = z$ so that the value of the integral in Eq. (13) is mainly determined by values of $\epsilon^{(n)}(z)$ in the vicinity of $z = x$.

If $K(x, z)$ and λ are positive and $b > a$, we have the inequality

$$\left|\epsilon^{(n)}(x) + \lambda \int K(x, z)\epsilon^{(n)}(z)\, dz\right| \leqq \epsilon^{(n)}_{\max}\left[1 + \lambda \int K(x, z)\, dz\right] \tag{14}$$

where $\epsilon^{(n)}_{\max}$ denotes the maximum amount of $\epsilon^{(n)}(x)$ for any value of x. Substituting Eqs. (6) and (14) in Eq. (13), we obtain

$$|\, k^{(n+1)}(x)\,| \leqq \epsilon^{(n)}_{\max} \quad \text{if} \quad K(x, z) > 0 \quad \text{and} \quad \lambda > 0, \tag{15}$$

i.e., the amount of the correction term for any value of x does not exceed the maximum amount of the deviation of the foregoing approximation from the true solution. In spite of Eq. (15), however, it may happen that $|\, \epsilon^{(n+1)}(x)\,| = |\, \epsilon^{(n)}(x) + k^{(n+1)}(x)\,|$ for some values of x is greater than $|\, \epsilon^{(n)}(x)\,|$ if the values of $\epsilon^{(n)}(x)$ are arbitrary. Thus a proof of convergence requires more elaborate considerations.

The following criterion of convergence is derived as an analogue to that for the solution of simultaneous linear algebraic equations according to von Mises and Pollaczek-Geiringer[3]. We assume that $\lambda > 0$ and $b > a$. From Eq. (13) it follows that

$$\begin{aligned} |\, \epsilon^{(n+1)}(x)\,| &= |\, \epsilon^{(n)}(x) + k^{(n+1)}(x)\,| \\ &\leqq |\, 1 + c(x)\,| \times |\, \epsilon^{(n)}(x)\,| + |\, c(x)\,| \lambda \int |\, K(x, z)\,| \times |\, \epsilon^{(n)}(z)\,|\, dz. \end{aligned} \tag{16}$$

We then calculate the integral of the amounts of $\epsilon^{(n+1)}(x)$,

$$\begin{aligned} \int |\, \epsilon^{(n+1)}(x)\,|\, dx \leqq &\int |\, 1 + c(x)\,| \times |\, \epsilon^{(n)}(x)\,|\, dx \\ &+ \int \left[|\, c(x)\,| \lambda \int |\, K(x, z)\,| \times |\, \epsilon^{(n)}(z)\,|\, dz\right] dx. \end{aligned} \tag{17}$$

Changing the sequence of integration in the second term of the right-hand member of Eq. (17), we obtain

$$\begin{aligned} \int |\, \epsilon^{(n+1)}(x)\,|\, dx \leqq &\int |\, 1 + c(x)\,| \times |\, \epsilon^{(n)}(x)\,|\, dx \\ &+ \int |\, \epsilon^{(n)}(z)\,| \lambda \left[\int |\, c(x)\,| \times |\, K(x, z)\,|\, dx\right] dz. \end{aligned} \tag{18}$$

To evaluate the second term of the right-hand member of Eq. (18), we have to calculate first the expression in brackets by integration with respect to the first variable of the kernel K. Then we have to integrate the product of this integral times $\epsilon^{(n)}$ with respect to the second variable of K. Interchanging the symbols x and z in the second term of the right-hand member of Eq. (18), we obtain

$$\begin{aligned}\int |\epsilon^{(n+1)}(x)|\,dx &\leqq \int |1 + c(x)| \times |\epsilon^{(n)}(x)|\,dx\,dx \\ &\quad + \int \left[|\epsilon^{(n)}(x)|\,\lambda \int |c(z)| \times |K(z, x)|\,dz \right] dx \qquad (19) \\ &= \int \left[|1 + c(x)| + \lambda \int |c(z)| \times |K(z, x)|\,dz \right] |\epsilon^{(n)}(x)|\,dx.\end{aligned}$$

We designate the maximum value of the expression in brackets in Eq. (19) for any value of x by M,

$$M \geqq \left[|1 + c(x)| + \lambda \int |c(z)| \times |K(z, x)|\,dz \right]. \qquad (20)$$

Combining Eqs. (19) and (20), we have

$$\int |\epsilon^{(n+1)}(x)|\,dx \leqq M \int |\epsilon^{(n)}(x)|\,dx. \qquad (21)$$

Consequently, the average of the amounts of the deviations $\epsilon^{(n)}(x), \epsilon^{(n+1)}(x), \cdots$ of the approximate solutions from the true solution converges toward zero if the value of M defined in Eq. (20) is a proper fraction. Convergence will be rapid if M is much less than unity. Since Eq. (21) represents only a necessary condition of convergence, however, it will happen that satisfactory convergence may also be found if the value of M is greater than unity.

4. Equations Resulting from Transformation of an Integral Equation into a Set of Simultaneous Linear Algebraic Equations. To apply the method suggested above, one may proceed either graphically or numerically. In the latter case, one has to transform the integrals in Eq. (7) into sums of values of the functions involved for specified values of $x = x_i$ and $z = x_j$,

$$\int K(x_i, z)\,dz = \Sigma_j \gamma_{ij} \qquad (22)$$

$$\int K(x_i, z)\psi^{(n)}(z)\,dz = \Sigma_j \gamma_{ij}\psi^{(n)}(x_j) \qquad (23)$$

where the coefficients γ_{ij} are determined by the values of $K(x_i, x_j)$. Hence we have instead of Eq. (7)

$$\psi^{(n+1)}(x_i) = \psi^{(n)}(x_i) + c_i\Delta^{(n)}(x_i) \qquad (24)$$

where

$$c_i = -1/[1 + \lambda\Sigma_j\gamma_{ij}] \qquad (25)$$

$$\Delta^{(n)}(x_i) = \psi^{(n)}(x_i) + \lambda\,\Sigma_j\gamma_{ij}\psi^{(n)}(x_j) - f(x_i). \qquad (26)$$

For comparison, we consider the formulae which result from transformations of integral Eq. (1) into a set of simultaneous linear algebraic equations with the unknowns $\varphi(x_i)$,

$$\Sigma_j a_{ij}\varphi(x_i) - f(x_i) = 0 \tag{27}$$

where $a_{ij} = \lambda\gamma_{ij}$ if $i \neq j$, and $a_{ij} = 1 + \lambda\gamma_{ij}$ if $i = j$. According to von Mises and Pollaczek-Geiringer[3], we may use the general iteration formula (24) with a conventional value

$$c_i' = -1/a_{ii} = -1/(1 + \lambda\gamma_{ii}) \tag{28}$$

instead of c_i according to Eq. (25). Then Eqs. (25) and (28) become virtually identical if the kernel has a high peak for $x = z$ and intervals between consecutive values of x_j are sufficiently large and thus the values of γ_{ii} equal nearly the sums $\Sigma\gamma_{ij}$.

On the other hand, if the number of values x_i is increased more and more, the values of individual coefficients γ_{ij} decrease. If $\gamma_{ii} \ll 1$, the value of c_i' approaches unity according to Eq. (28). Then, with $\psi^{(0)}(x) = f(x)$, the Liouville-Neumann series results.

5. Modification of the Foregoing Iteration Method. According to Seidel,[4] the iteration process for the solution of a set of simultaneous linear algebraic equations (27) with a conventional value of $c_i' = -1/a_{ii}$ can be improved by introducing values of the $(n + 1)^{\text{th}}$ approximation of the unknowns into the expression for $\Delta^{(n)}(x_i)$ as far as these values have been calculated before. Accordingly we modify the procedure described above. We define a function

$$\Delta^{*(n)}(x_i) = \psi^{(n)}(x_i) + \lambda\sum_{j=1}^{j=i-1}\gamma_{ij}\psi^{(n+1)}(x_i) + \lambda\sum_{j=i}^{j=\nu}\gamma_{ij}\psi^{(n)}(x_j) - f(x_i) \tag{29}$$

where ν is the total number of specified values x_i. In view of Eq. (2), one may also write

$$\Delta^{*(n)}(x_i) = \Delta^{*(n-1)}(x_i) + k^{(n)}(x_i) + \lambda\sum_{j=1}^{j=i-1}\gamma_{ij}k^{(n+1)}(x_i) + \lambda\sum_{j=i}^{j=\nu}\gamma_{ij}k^{(n)}(x_i). \tag{30}$$

The computation of the right-hand side of Eq. (30) requires less labor especially when one is close to the final solution and some values of $k^{(n)}(x_i)$ may be negligible, but Eq. (29) has the advantage that eventual errors of computation are eliminated in successive approximations.

Instead of Eq. (24), we have

$$\psi^{(n+1)}(x_i) = \psi^{(n)}(x_i) + c_i\Delta^{*(n)}(x_i). \tag{31}$$

To obtain a criterion of convergence, we may apply the same kind of reasoning as Seidel[4] and von Mises and Pollaczek-Geiringer[3] have used for the conventional value of c_i' indicated in Eq. (28). We assume that $K(x, z)$ is a symmetrical function with respect to x and z, i.e., $K(x, z) = K(z, x)$. We then define a function

$$\begin{aligned} F[\varphi(x_1), \varphi(x_2), \cdots, \varphi(x_\nu)] &= \tfrac{1}{2}\{\textstyle\sum_i [\varphi(x_i)]^2 + \lambda\sum_i\sum_j \gamma_{ij}\varphi(x_i)\varphi(x_j)\} - \sum_i f(x_i)\varphi(x_i) \\ &= Q - \textstyle\sum_i f(x_i)\varphi(x_i) \end{aligned} \tag{32}$$

where the $\varphi(x_i)$'s can assume arbitrary values. The set of linear algebraic equations indicated in Eq. (27) is identical with

$$\partial F/\partial\varphi(x_i) = 0 \qquad (i = 1, 2, \cdots, \nu). \tag{33}$$

Seidel and von Mises and Pollaczek-Geiringer have shown that Eqs. (33) determine a minimum value of the function F and that only one minimum of F exists if the value of Q defined in Eq. (32) is positive definite. Solving the set of linear algebraic equations indicated in Eq. (27) becomes, therefore, identical with determining the variables of F which correspond to a minimum value of F. The iteration process is convergent if F decreases in each step. To prove convergence, we calculate the change ΔF of the function F in an individual step of the iteration method indicated in Eqs. (29) and (31), values of $\psi^{(n+1)}$ and $\psi^{(n)}$, respectively, being substituted for values of φ in Eq. (32). Thus, in view of $K(x_i, x_j) = K(x_j, x_i)$,

$$\Delta F = \tfrac{1}{2}(1 + \lambda\gamma_{ii})\{[\psi^{(n+1)}(x_i)]^2 - [\psi^{(n)}(x_i)]^2\} + \{\lambda\textstyle\sum_{j=1}^{j=i-1}\gamma_{ij}\psi^{(n+1)}(x_j) \tag{34}$$
$$+ \lambda\textstyle\sum_{j=i+1}^{j=\nu}\gamma_{ij}\psi^{(n+1)}(x_j) - f(x_i)\}[\psi^{(n+1)}(x_i) - \psi^{(n)}(x_i)].$$

From Eqs. (29) and (31) it follows that

$$\lambda\textstyle\sum_{j=1}^{j=i-1}\gamma_{ij}\psi^{(n+1)}(x_j) + \lambda\sum_{j=i+1}^{j=\nu}\gamma_{ij}\psi^{(n)}(x_j) - f(x_i) = \Delta^{*(n)}(x_i)$$
$$- (1 + \lambda\gamma_{ii})\psi^{(n)}(x_i) = [\psi^{(n+1)}(x_i) - \psi^{(n)}(x_i)]/c_i - (1 + \lambda\gamma_{ii})\psi^{(n)}(x_i). \tag{35}$$

Substitution in Eq. (34) yields

$$\Delta F = -\tfrac{1}{2}[-1 - \lambda\gamma_{ii} - 2/c_i]\,\{\psi^{(n+1)}(x_i) - \psi^{(n)}(x_i)\}^2. \tag{36}$$

The function F decreases in each step if the factor in brackets is positive. Consequently, convergence of iteration based on Eqs. (6), (29), and (31) is ascertained if (1) the kernel $K(x, z)$ is symmetric with respect to x and z, (2) the value of Q defined in Eq. (32) is positive definite, and (3) the factor $(-1 - \lambda\gamma_{ij} - 2/c_i) = 1 + 2\lambda\Sigma\gamma_{ij} - \lambda\gamma_{ii}$ according to Eq. (25) is positive for any value of x_i.

One can show that the decrease of F in a single step has a maximum value if c_i is replaced by $c_i' = -1/q_{ii} = -1/(1 + \lambda\gamma_{ii})$ according to Eq. (28). The most rapid convergence for the whole set of values x_i is, however, a different problem. To this end, the decrease of F after a sequence of individual steps from $i = 1$ to $i = \nu$ should have a maximum value. Considerations omitted here show that it is not possible to determine values of c_i so that for any values of $\psi^{(n)}(x_i)$ a maximum decrease of F results. Considerations in Section 2 lead to the conclusion that use of the special value of c_i as determined by Eq. (25) is, in general, advantageous if $K(x, z)$ has a pronounced maximum for $x = z$. As pointed out above, the exact solution will be reached immediately if the kernel vanishes except for $x = z$, so that integral Eq. (1) becomes a linear equation in $\varphi(x)$. Since this is not the case in practical problems, one may therefore make a compromise and choose a value between the two values indicated in Eqs. (25) and Eq. (28), but no general rule for such a compromise can be given.

6. Illustrative Example. The foregoing considerations may be illustrated by the integral equation

$$\varphi(x) = 1 - \lambda \int_0^1 [-\ln|x^2 - z^2|]\varphi(z)\,dz. \tag{37}$$

This integral equation results from the boundary problem of third kind of the Laplace differential equation for the potential in an $x - y$-plane when the sum of the potential and the product of the potential gradient times a parameter along the line $-1 < x < +1, y = 0$ is given as a constant value, and the potential gradient along this line, proportional to the density of electrical charge and proportional to $\varphi(x)$ in Eq. (37), is to be calculated. For illustration, we assume $\lambda = 2$.

TABLE 1

Successive approximations for integral equation (37) with $\lambda = 2$

i	x_i	$\psi^{(1)}$	$k^{(1)}$	$\psi^{(2)}$	$k^{(2)}$	$\psi^{(3)}$	$k^{(4)}$	$\psi^{(4)}$
1	0.95	0.3711	+0.0416	0.4127	+0.0145	0.4272	+0.0047	0.4319
2	0.85	0.3037	+0.0138	0.3175	+0.0060	0.3235	+0.0029	0.3264
3	0.75	0.2678	+0.0036	0.2714	+0.0024	0.2738	+0.0013	0.2715
4	0.65	0.2450	−0.0015	0.2435	+0.0003	0.2432	+.00007	0.2439
5	0.55	0.2294	−0.0044	0.2250	−0.0008	0.2242	±0.0000	0.2242
6	0.45	0.2183	−0.0061	0.2122	−0.0016	0.2106	−0.0004	0.2102
7	0.35	0.2105	−0.0071	0.2034	−0.0022	0.2012	−0.0006	0.2006
8	0.25	0.2052	−0.0074	0.1978	−0.0026	0.1952	−0.0009	0.1943
9	0.15	0.2018	−0.0073	0.1945	−0.0028	0.1917	−0.0011	0.1906
10	0.05	0.2002	−0.0067	0.1935	−0.0028	0.1907	−0.0013	0.1894

If one tries to use the Liouville-Neumann series, one has $\psi^{(0)}(x) = 1$ and subsequently $\psi^{(1)}(0) = 1 - 2\lambda = -3$ and $\psi^{(1)}(1) = 1 - 0.6\lambda = -0.2$, whereas the true values of $\varphi(x)$ are positive throughout the whole range from $x = 0$ to $x = 1$. The Liouville-Neumann series is therefore not helpful for $\lambda = 2$.

Instead, a first approximation may be obtained with the help of Eq. (8), which is based on the value of $c(x)$ suggested in Eq. (6). To compute further approximations, we write in accordance with Eqs. (22) and (23)

$$\gamma_{ij} = \int_{\nu h - jh}^{\nu h + h - jh} \{-\ln[(\nu h + \tfrac{1}{2}h - hi)^2 - \zeta^2]\}\,d\zeta \tag{38}$$

where $\nu h + \frac{1}{2}h - ih = x_i$ and ζ is an auxiliary variable of integration. For a first approach, a value of $h = 0.1$ may be chosen so that $x_i = 0.95, 0.85, \cdots, 0.05$. Values of the first, second, third, and fourth approximation and values of $k^{(n)}(x_i)$ calculated with the aid of Eqs. (6), (29), (31), and (38) are compiled in Table I. To obtain more accurate results, one has to decrease the width of intervals, preponderatingly at $x \sim 1$, where the function $\varphi(x)$ changes very rapidly.

For small values of λ, Eq. (37) may be solved with the aid of the Liouville-Neumann series. After having calculated the numerical values of the iterated kernels, one obtains the solution of $\varphi(x)$ as a power series in λ, which gives the

possibility of calculating $\varphi(x)$ at once for any value of λ. Even for small values of λ, however, it may be expedient to calculate $\varphi(x)$ for selected values of λ by means of the method suggested above, and herefrom values of $\varphi(x)$ for intermediate values of λ by interpolation, since the new method converges more rapidly than the Liouville-Neumann series. For $\lambda = 0.25$, e.g., it follows from Eqs. (3), (8), (5), and (6) that

$$\varphi(x = 0.95) = 0.825 + 0.009 \cdots$$

$$\varphi(x = 0.05) = 0.669 - 0.005 \cdots,$$

whereas the Liouville-Neumann series gives

$$\varphi(x = 0.95) = 1 - 0.212 + 0.064 - 0.024 + 0.010 - \cdots$$

$$\varphi(x = 0.05) = 1 - 0.500 + 0.234 - 0.107 + 0.048 - \cdots$$

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STREAM FUNCTIONS FOR THREE-DIMENSIONAL FLOWS

By J. H. Giese

It is well known that plane or axisymmetric steady, non-viscous, adiabatic flows with vorticity can be described completely by means of a single stream function. The fact that steady three-dimensional flows with vorticity can be described by means of two stream functions has received less attention, perhaps because the formidable appearance of the system of partial differential equations for these functions makes their usefulness doubtful. In this note it is shown that these general stream functions have properties analogous to those of the familiar special cases, and that they can be used to verify a variational principle for three-dimensional flows.

Steady adiabatic flow of a perfect fluid is governed by the system [7][1]

$$\mathbf{u} \times (\nabla \times \mathbf{u}) = -\tfrac{1}{2}(c^2 - u^2)\nabla \log f \tag{1}$$

$$\nabla \cdot (\rho \mathbf{u}) = 0 \tag{2}$$

$$p = f\rho^\gamma, \qquad \log f = S/c_v \tag{3}$$

$$\mathbf{u} \cdot \nabla f = 0 \tag{4}$$

$$\tfrac{1}{2}u^2 + \gamma p/(\gamma - 1)\rho = \tfrac{1}{2}c^2, \qquad \mathbf{u} \cdot \nabla c = 0 \tag{5}$$

where $\mathbf{u}$ is the velocity, p the pressure, ρ the density, S the specific entropy, $\gamma = c_p/c_v$, and c_p and c_v are the specific heats of the fluid at constant pressure and volume, respectively. The dependent variables $\mathbf{u}$, p, ρ, and f can always be redefined [3] so that

$$c = 1 \tag{6}$$

It will be assumed hereafter that this simplification has been made. Now (3) and (5) are equivalent to

$$\begin{aligned} p &= f^{-1/(\gamma-1)}[\tfrac{1}{2}(\gamma - 1)\gamma^{-1}(1 - u^2)]^{\gamma/(\gamma-1)} \\ \rho &= f^{-1/(\gamma-1)}[\tfrac{1}{2}(\gamma - 1)\gamma^{-1}(1 - u^2)]^{1/(\gamma-1)} \end{aligned} \tag{7}$$

By (4) and (7), (2) implies

$$\nabla \cdot (1 - u^2)^{1/(\gamma-1)}\mathbf{u} = 0 \tag{8}$$

an equation previously obtained by Crocco [2].

By (2) and (8) $\rho\mathbf{u}$ and $(1 - u^2)^{1/(\gamma-1)}\mathbf{u}$ are solenoidal vectors. Hence they can be represented as vector products of two gradients [6]. In particular, we may write

$$\rho\mathbf{u} = \nabla\psi_1 \times \nabla\psi_2 \tag{9}$$

for some two functions ψ_1 and ψ_2. Since

$$\mathbf{u} \cdot \nabla\psi_\lambda = 0, \qquad \lambda = 1, 2 \tag{10}$$

[1] Bracketed numbers designate references listed at the end.

$\mathbf{u}$ is tangent to the two families of surfaces ψ_λ = constant. Hence these are stream surfaces, i.e. no fluid crosses them, and *the intersection of two surfaces* ψ_1 = constant *and* ψ_2 constant *is a streamline.*

If we choose $\psi_1 = -z$ and $\partial\psi_2/\partial z = 0$, then $\psi_2 = \psi(x, y)$ reduces to the stream function for plane flow parallel to the xy-plane. Likewise, if we choose $\psi_1 = -\arctan y/x$ and $\psi_2 = \psi(r, z)$, where $r^2 = x^2 + y^2$, then ψ is the stream function for flow axisymmetric about the z-axis. For these special cases the mass m of fluid that crosses per second the region cut out of any surface by a stream tube bounded by the surface $\psi_1 = a$, $\psi_1 = b$, $\psi_2 = c$, $\psi_2 = d$ (where $|\, b - a \,| \leqq 2\pi$ in the axisymmetric case) is

$$m = |\, b - a \,| \cdot |\, d - c \,| \tag{11}$$

This can be generalized as follows. Let $\mathbf{r} = (x, y, z)$ denote the coordinate vector. In the region occupied by the fluid let $\mathbf{r} = \mathbf{r}(\alpha, \beta)$ be parametric equations of a surface Σ other than a stream surface. Then $\mathbf{n} = \partial\mathbf{r}/\partial\alpha \times \partial\mathbf{r}/\partial\beta$ is a vector normal to Σ, and $d\sigma = |\, n \,|\, d\alpha\, d\beta$ is the surface element of Σ. On Σ let R be a finite region bounded by a simply closed curve K defined by $\alpha = \alpha(\epsilon)$, $\beta = \beta(\epsilon)$. Assume that ψ_1 and ψ_2 are defined everywhere in R. Then the mass $m(R)$ of fluid crossing R per second is

$$\begin{aligned} m(R) &= \left| \iint_R (\rho \mathbf{u} \cdot \mathbf{n}/|\, n \,|) d\sigma \right| \\ &= \left| \iint_R (\nabla\psi_1 \times \nabla\psi_2) \cdot (\partial\mathbf{r}/\partial\alpha \times \partial\mathbf{r}/\partial\beta)\, d\alpha\, d\beta \right| \\ &= \left| \iint_R \partial(\psi_1, \psi_2)/\partial(\alpha, \beta)\, d\alpha\, d\beta \right| \end{aligned} \tag{12}$$

In the $\psi_1\psi_2$-plane let C be the closed curve $\psi_\lambda = \psi_\lambda(\mathbf{r}(\alpha(\epsilon), \beta(\epsilon))$, the image of K. Assume that the map of K onto C is biunique. Then C is also simply closed and bounds a finite region Q, the image of R. Now (12) becomes

$$m(R) = \left| \iint_Q d\psi_1\, d\psi_2 \right| \tag{13}$$

i.e. $m(R)$ *is the area of* Q. In the special case in which Q is bounded by $\psi_1 = a$, $\psi_1 = b$, $\psi_2 = c$, $\psi_2 = d$ (13) reduces to (11).

By another choice of ψ_λ we may write

$$(1 - u^2)^{1/(\gamma-1)}\mathbf{u} = \nabla\psi_1 \times \nabla\psi_2 \tag{14}$$

For these functions (10) still holds, so ψ_λ = constant are still stream surfaces. In the plane and axisymmetric cases these stream funcitons are of the type introduced by Crocco [2]. Now assume that $\mathbf{u}$ is non-null and $u^2 \neq 1$. Then by (14) ψ_1 and ψ_2 are independent, while by (4) and (10)

$$f = f(\psi_1, \psi_2) \tag{15}$$

Lin and Rubinov [4, 5] have shown that the variational equations for

$$I = \iint_{\Sigma} (p + \rho u^2)(2\pi y)^{\epsilon}\, dy\, dx \tag{16}$$

are the equations for plane ($\epsilon = 0$) or axisymmetric ($\epsilon = 1$) flows by using the type of ψ_2 that corresponds to (14) when ψ_1 is given one of the special forms mentioned above. This result can easily be extended to three-dimensional flows. The method used here to verify the variational principle differs somewhat from that of Bateman [1], who employed another, more general representation for the velocity, suitable for non-steady flow, and had more dependent variables in the integrand of his variational integral. Let

$$F = 2\gamma[2\gamma/(\gamma - 1)]^{1/(\gamma-1)}(p + \rho u^2) \tag{17}$$

By (7) and (17)

$$F(\psi, \nabla\psi) = f^{-1/(\gamma-1)}[(\gamma - 1)\,(1 - u^2)^{\gamma/(\gamma-1)} + 2\gamma(1 - u^2)^{1/(\gamma-1)}u^2] \tag{18}$$

where by (14) u^2 is defined implicitly by

$$(1 - u^2)^{2/(\gamma-1)}u^2 = (\nabla\psi_1 \times \nabla\psi_2)^2 \tag{19}$$

(in fact, assuming $(\nabla\psi_1 \times \nabla\psi_2)^2 < [2/(\gamma + 1)]^{2/(\gamma-1)}(\gamma - 1)/(\gamma + 1)$, (19) has in the interval (0, 1) two roots, one sub-, the other supersonic). The first variations of

$$I(\psi) = \iiint_{V} F(\psi, \nabla\psi)\, dv \tag{20}$$

where V is a finite three-dimensional region with boundary Σ, are

$$\delta_\lambda I(\psi) = \iiint_{V} (\partial F/\partial\psi_\lambda - \nabla\cdot\partial F/\partial\nabla\psi_\lambda)\delta\psi_\lambda\, dv + \iint_{\Sigma} (\partial F/\partial\nabla\psi_\lambda)\cdot\mathbf{n}\delta\psi_\lambda\, d\sigma \tag{21}$$

where dv and $d\sigma$ are the elements of volume and surface of V and Σ, and $\mathbf{n}$ is the outwardly directed unit vector normal to Σ. The corresponding Euler equations are

$$\nabla\cdot\partial F/\partial\nabla\psi_\lambda - \partial F/\partial\psi_\lambda = 0 \tag{22}$$

By (18), (19), and (14)

$$\partial F/\partial\nabla\psi_\lambda = (-1)^\lambda 2\gamma f^{-1/(\gamma-1)}\mathbf{u} \times \nabla\psi_{\lambda+1} \tag{23}$$

where $\lambda + 1$ is to be reduced mod 2. Hence (22) becomes

$$(-1)^{\lambda+1}\nabla\psi_\lambda\cdot(\nabla \times \mathbf{u}) = -\tfrac{1}{2}\gamma^{-1}(1 - u^2)^{\gamma/(\gamma-1)}\partial \log f/\partial\psi_{\lambda+1} \tag{24}$$

which implies

$$(\nabla\psi_1 \times \nabla\psi_2) \times (\nabla \times \mathbf{u}) = -\tfrac{1}{2}\gamma^{-1}(1 - u^2)^{\gamma/(\gamma-1)}\nabla \log f \tag{25}$$

By (14), (25) implies (1). Since (14) and (7) have been used to define $\mathbf{u}$, p, and ρ, and since f has been assumed to be of the form (15), it suffices to remark that

these imply (3), (4), (5), and (8). By (21) natural boundary conditions for ψ_λ are such that

$$\iint_\Sigma (\partial F/\partial \nabla\psi_\lambda)\cdot \mathbf{n}\delta\psi_\lambda \, d\sigma = 0 \tag{26}$$

If the values of $\mathbf{u}$ and f have been prescribed on a two-dimensional region Σ' of Σ which is not a stream surface, suitable boundary values of ψ_λ in Σ' can be found as described below. Then in Σ' we may assume $\delta\psi_\lambda = 0$ for $\lambda = 1, 2$. If Σ contains a two-dimensional region Σ'' which is a stream surface, then the boundary values of the ψ_λ can always be adjusted so that one or the other, e.g. ψ_1, is constant on Σ''. On Σ'' then $\delta\psi_1 = 0$, so that $(\mathbf{u}\cdot\nabla\psi_2 \times \mathbf{n})\delta\psi_1 = 0$, while $(\mathbf{u}\cdot\nabla\psi_1 \times \mathbf{n})\delta\psi_2 = 0$ for arbitrary $\delta\psi_2$ since $\nabla\psi_1 \times \mathbf{n} = 0$. Hence by (23) the surface integrals (26) will vanish on Σ'' as well as on Σ'.

From (1), (14), and (19), after considerable computation, we obtain for ψ_λ the system of partial differential equations

$$\begin{aligned}[\mathbf{u}\cdot(\nabla\nabla\psi_\lambda)\cdot\mathbf{u} + \tfrac{1}{2}\gamma^{-1}(1-u^2)\textstyle\sum_\mu(\partial \log f/\partial\psi_\mu)\nabla\psi_\mu\cdot\nabla\psi_\lambda(1-M^2)\,(1-u^2)^{1/(\gamma-1)} \\ + (\mathbf{u}\times\nabla\psi_2)\cdot(\nabla\nabla\psi_1) - (\mathbf{u}\times\nabla\psi_1)\cdot(\nabla\nabla\psi_2)]\cdot\nabla\psi_\lambda = 0\end{aligned} \tag{27}$$

where the Mach number M is defined by $M^2 = 2u^2/(\gamma - 1)\,(1 - u^2)$. Since for inviscid flow vorticity is generated when initially irrotational flow passes through a curved shock wave (disregarding certain developable shock wave shapes), this suggests the problem, to determine the flow behind a *given* shock surface, assuming the flow ahead of the shock to be known. This leads, in turn, to the more general initial value problem, to determine a flow such that on a given initial surface Σ (not a stream surface) defined by parametric equations $\mathbf{r} = \mathbf{r}_0(\alpha, \beta)$ the functions $\mathbf{u}$ and f assume prescribed values $\mathbf{u}(\mathbf{r}_0(\alpha, \beta)) = \mathbf{u}_0(\alpha, \beta)$ and $f(\mathbf{r}_0(\alpha, \beta)) = f_0(\alpha, \beta)$. If an attempt were made to solve this problem by means of the system (27), initial values $\psi_\lambda(\mathbf{r}_0(\alpha, \beta)) = \psi_{\lambda 0}(\alpha, \beta)$ would be required. These can be found as follows. First form the strip conditions

$$\begin{aligned}\partial\psi_{\lambda 0}/\partial\alpha &= (\nabla\psi_\lambda)_0\cdot\partial\mathbf{r}_0/\partial\alpha \\ \partial\psi_{\lambda 0}/\partial\beta &= (\nabla\psi_\lambda)_0\cdot\partial\mathbf{r}_0/\partial\beta\end{aligned} \tag{28}$$

and also insist that (10) hold on Σ so that

$$(\nabla\psi_\lambda)_0\cdot\mathbf{u}_0 = 0 \tag{29}$$

Then

$$\begin{aligned}(\nabla\psi_\lambda)_0(\partial\mathbf{r}_0/\partial\alpha \times \partial\mathbf{r}_0/\partial\beta\cdot\mathbf{u}_0) = \\ (\partial\psi_{\lambda 0}/\partial\alpha)\,(\partial\mathbf{r}_0/\partial\beta \times \mathbf{u}_0) - (\partial\psi_{\lambda 0}/\partial\beta)\,(\partial\mathbf{r}_0/\partial\alpha \times \mathbf{u}_0)\end{aligned} \tag{30}$$

If we also demand that (14) hold on Σ, then by (30) and (14)

$$\partial(\psi_{10}, \psi_{20})/\partial(\alpha, \beta) = (1 - u_0^2)^{1/(\gamma-1)}(\partial\mathbf{r}_0/\partial\alpha \times \partial\mathbf{r}_0/\partial\beta\cdot\mathbf{u}_0) \tag{31}$$

Hence, if the family of curves $\psi_{10} =$ constant is chosen arbitrarily on Σ, the family $\psi_{20} =$ constant must be chosen to satisfy (31). Invert $\psi_{\lambda 0} = \psi_{\lambda 0}(\alpha, \beta)$

to obtain $\alpha = \alpha(\psi_{10}, \psi_{20})$, $\beta = \beta(\psi_{10}, \psi_{20})$. Substitute these in $f_0(\alpha, \beta)$ to obtain $f_0(\alpha, \beta) = f(\psi_{10}, \psi_{20})$. The function $f(\psi_1, \psi_2)$ so defined is to be used in (1), (7), (15), (18), (27), etc.

The question arises, whether the values of both of ψ_λ can be preserved across a shock. Suppose they can. The fact that the value of $\rho\mathbf{u}\cdot\mathbf{n}$ is preserved, where $\mathbf{n}$ is a unit vector normal to the shock, together with (7) and (31) would then imply that the value of f is preserved. In general, this is false. Thus we *cannot*, in general, preserve the values of both of the stream functions in (14), though we can preserve those of ψ_1 alone. Similar reasoning, using an analog to (31) with ρ_0 substituted for $(1 - u^2)^{1/(\gamma-1)}$ shows that the values of both of the stream functions in (9) *can* be preserved at a shock wave. Of course, this is an obvious consequence of the relation between these functions and the rate of mass flow.

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NOTE ON LEVINSON'S EXISTENCE THEOREM FOR FORCED PERIODIC SOLUTIONS OF A SECOND ORDER DIFFERENTIAL EQUATION

By C. E. Langenhop

In his paper "On the Existence of Periodic Solutions for Second Order Differential Equations with a Forcing Term",[1] Levinson proves the existence of a periodic solution for the equation

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = e(t), \tag{1.0}$$

or the equivalent system

$$\dot{x} = v, \qquad \dot{v} = -f(x, v)v - g(x) + e(t). \tag{1.0'}$$

His assumptions concerning $g(x)$ are essentially

$$\lim_{|x|\to\infty} |\, g(x)\,| = \infty, \qquad xg(x) > 0 \text{ when } |\, x\,| > a, \tag{1.1}$$

and

$$\lim_{x\to\infty} g(x)/G(x) = 0, \qquad G(x) = \int_0^x g(t)dt. \tag{1.2}$$

The proof of the existence of a periodic solution of (1.0) of period L makes use of the fact that for large $|\, x\,|$ and $|\, v\,|$ a physical system governed by this equation with the conditions Levinson makes on $g(x), f(x)$ and $e(t)$ would dissipate energy and $|\, x\,|$ and $|\, v\,|$ would tend to become smaller. Intuitively then the condition (1.2) seems unnecessary, for if, say, $g(x) = xe^{x^2}$ this condition is violated, although such a $g(x)$ represents an even stronger restoring force than, for example, $g(x) = x$. The purpose of this note is to show how at one place Levinson's proof may be altered so as to prove the theorem with (1.2) replaced by

$$\limsup_{x\to\infty} g(x)/x = \infty. \tag{1.2'}$$

The conditions (1.2) and (1.2') cover all possibilities for a function $g(x)$ satisfying (1.1). Suppose $\limsup_{x\to\infty} g(x)/x \neq \infty$. Then there exists an $N > 0$ and $X > 0$ such that $g(x) < Nx$ when $x > X$.

Now by L'Hospital's rule

$$\lim_{x\to\infty} Nx/G(x) = \lim_{x\to\infty} N/g(x) = 0,$$

and since for $x > X$ $0 < g(x)/G(x) < Nx/G(x)$ such a $g(x)$ would satisfy (1.2). Although the main part of the proof using either (1.2) or (1.2') is the same, there seems to be no uniform way of treating all $g(x)$ satisfying (1.1).

Levinson constructs a simple closed curve C in the (x, v) plane which he

[1] N. Levinson, this journal, Vol. 22 (1943), p. 41.

denotes by $P_1P_2 \cdots P_{11}P_{12}P_1$.[2] If we replace (1.2) by (1.2′), then we alter his construction by taking, instead of a horizontal line for P_4P_5, the arc of the solution of the system

$$\dot{x} = v, \qquad \dot{v} = -mv - g(x) - E \tag{1.3}$$

or what is the same, the solution of

$$\frac{dv}{dx} = \frac{-mv - g(x) - E}{v} = h(x, v) \tag{1.4}$$

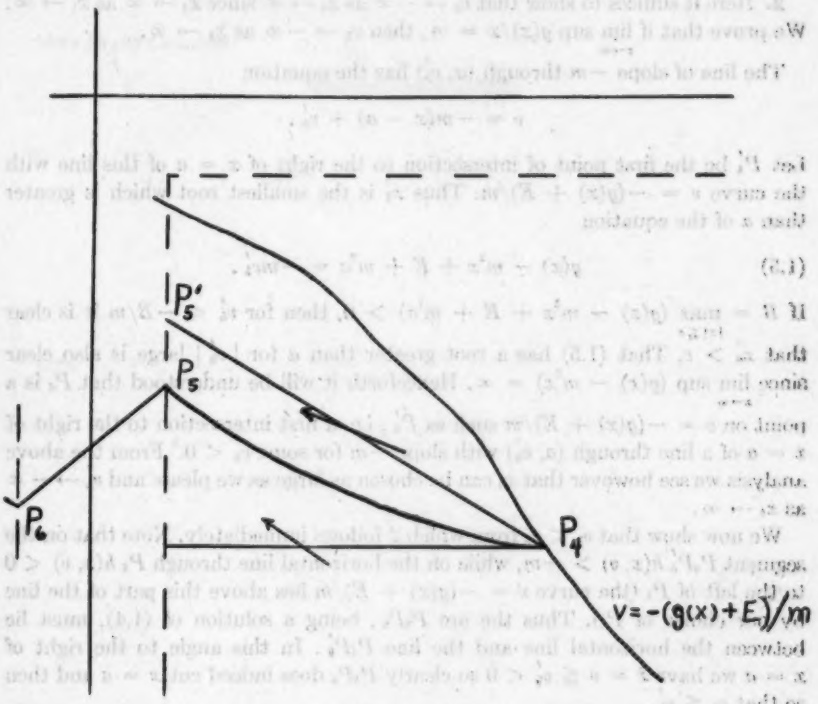

FIGURE 1

which proceeds to the left from P_4 and cuts the line $x = a$. See Fig. 1. (We prove later that this arc does indeed cut this line.) If this point of intersection with $x = a$ is denoted by P_5, then the remainder of the curve C is as prescribed in Levinson's paper.

We must now show: 1. *that the solutions of* (1.0′) *cut the new arc* P_4P_5 *in the desired direction*, 2. *that as* $x_1 \to \infty$, $v_5 \to -\infty$ (so that P_7 can be taken sufficiently far to the left as is required in another part of Levinson's proof) and 3. *that for* x_1 *sufficiently large* $u_{12} - u_1 < 0$. These we proceed to establish.

[2] See Levinson's paper for the details of this construction and for the definitions of any symbols not herein defined.

1. At points of the arc P_4P_5 of Fig. 1 we have for the solutions of (1.0′)

$$\frac{\dot{v}}{\dot{x}} = \frac{-f(x, v)v - g(x) + e(t)}{v} < \frac{-mv - g(x) - E}{v}$$

the slope of P_4P_5. Thus since $\dot{x} = v < 0$ on this arc, the solutions of (1.0′) do cut P_4P_5 so as to go from the exterior of C to the interior as t increases.

2. Here it suffices to show that $v_5 \to -\infty$ as $x_4 \to \infty$ since $x_4 \to \infty$ as $x_1 \to \infty$. We prove that if $\limsup_{x\to\infty} g(x)/x = \infty$, then $v_5 \to -\infty$ as $x_4 \to \infty$.

The line of slope $-m$ through (a, v_5') has the equation

$$v = -m(x - a) + v_5'.$$

Let P_4' be the first point of intersection to the right of $x = a$ of this line with the curve $v = -(g(x) + E)/m$. Thus x_4' is the smallest root which is greater than a of the equation

$$g(x) - m^2x + E + m^2a = -mv_5'. \tag{1.5}$$

If $B = \max_{|x|\le c} (g(x) - m^2x + E + m^2a) > 0$, then for $v_5' < -B/m$ it is clear that $x_4' > c$. That (1.5) has a root greater than a for $|v_5'|$ large is also clear since $\limsup_{x\to\infty} (g(x) - m^2x) = \infty$. Henceforth it will be understood that P_4 is a point on $v = -(g(x) + E)/m$ such as P_4', i.e. a first intersection to the right of $x = a$ of a line through (a, v_5') with slope $-m$ for some $v_5' < 0$.[3] From the above analysis we see however that x_4 can be chosen as large as we please and $v_5' \to -\infty$ as $x_4 \to \infty$.

We now show that $v_5 < v_5'$ from which 2 follows immediately. Note that on the segment P_4P_5' $h(x, v) > -m$, while on the horizontal line through P_4 $h(x, v) < 0$ to the left of P_4 (the curve $v = -(g(x) + E)/m$ lies above this part of the line by our choice of P_4). Thus the arc P_4P_5, being a solution of (1.4), must lie between the horizontal line and the line P_4P_5'. In this angle to the right of $x = a$ we have $\dot{x} = v \le v_5' < 0$ so clearly P_4P_5 does indeed cut $x = a$ and then so that $v_5 \le v_5'$.

3. Finally we show that for x_1 large enough $u_{12} - u_1 < 0$. The differences $u_r - u_{r-1}$, $r \ne 5$ are the same as in Levinson's paper. Now, however, by (1.4)

$$u_5 - u_4 = \int_{P_4}^{P_5} v\,dv + g(x)\,dx = \int_a^{x_4} (E + mv)\,dx < (E + mv_5)(x_4 - a)$$

if on P_4P_5 we always have $v \le v_5$. This is certainly the case since P_4P_5 lies below

[3] Levinson tacitly makes a similar assumption concerning the horizontal segment through P_4 for if this line were to cut the curve $v = -(g(x) + E)/m$ at some x such that $a < x < x_4$, then it would not be true that $\dot{v}$ of (1.0′) > 0 at all points of the line to the left of P_4.

the line P_4P_5' which in turn lies below the curve $v = -(g(x) + E)/m$, so on P_4P_5 $dv/dx = h(x, v) \leqq 0$. We have then

$$u_{12} - u_1 = \sum_{r=2}^{12} (u_r - u_{r-1})$$

$$\leqq (E + mv_5)(x_4 - a - 8aM/m) + 32a^3M^2 + 4a^3M + 2a^2 + 8aME/m$$

from which it is clear, since $v_5 \to -\infty$ as $x_4 \to \infty$, that for large enough x_4 we have $u_{12} - u_1 < 0$.

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THE INTEGRAL $\int_0^\infty y^3 \exp(-y^2 + ix/y)\, dy$

By U. E. Kruse and N. F. Ramsey

1. Introduction. In various problems of theoretical physics there arise definite integral functions which can be reduced to one of the following forms:

$$I(x) = \int_0^\infty \exp(-y^2) y^3 \cos(x/y)\, dy, \tag{1}$$

$$K(x) = \int_0^\infty \exp(-y^2) y^3 \sin(x/y)\, dy, \tag{2}$$

or

$$I(x) + iK(x) = \int_0^\infty y^3 \exp(-y^2 + ix/y)\, dy. \tag{3}$$

In particular the average over the neutron beam velocity distribution of a trigonometric function of the time a neutron spends in a region of space reduces to an expression of the above type.[1,2] Similarly expressions of the above type arise in the theory of the molecular beam magnetic resonance method.[3–5] The above expressions are not reducible to any of the standard tabulated definite integrals because of the occurrence of the y both as y^2 and as $1/y$ in the argument of the exponential. Likewise because of the oscillatory character of the integrand for large x and small y, numerical integration is difficult. Although a convergent series expansion of the integrals is possible for $x < 3$ and an asymptotic series can be obtained for larger values of x, the asymptotic series is sufficiently awkward for numerical calculation that there is need of a direct tabulation of the functions $I(x)$ and $K(x)$.

2. Series Expansions. Zahn[1] and Laporte[2] have discussed a function

$$\phi_1(x) = \int_0^\infty \exp(-y - x/\sqrt{y}) y\, dy \tag{4}$$

which is related to the above functions by the equations

$$I(x) = 1/2 \text{ Re } \{\phi_1(ix)\} \qquad K(x) = -1/2 \text{ Im}\{\phi_1(ix)\}. \tag{5}$$

For values of $x < 3$ Zahn has developed a convergent series which in the present case becomes

$$I(x) = \tfrac{1}{2}[\textstyle\sum_{n=0}^\infty \alpha_{2n} x^{2n} - x^4 \log x \sum_{n=0}^\infty \beta_{2n} x^{2n}] \tag{6}$$

[1] C. T. Zahn, Phys. Rev. **52**, 67 (1937).
[2] O. Laporte, Phys. Rev. **52**, 72 (1937).
[3] I. I. Rabi, Phys. Rev. **51**, 625 (1937).
[4] H. C. Torrey, Phys. Rev. **59**, 293 (1941).
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$$K(x) = \tfrac{1}{2}[\textstyle\sum_{n=0}^{\infty} \alpha_{2n+1} x^{2n+1} + (\pi/2) x^4 \sum_{n=0}^{\infty} \beta_{2n} x^{2n}] \qquad (7)$$

where numerical values for α_i and β_i are given in Table 1.

Because of the slow convergence of the above series for larger values of x, it is more convenient to use an asymptotic expansion due to Laporte[2]

$$\phi_1(x) = 2(\pi/3)^{1/2}(x/2) \exp -[(3/2)(x/2)^{2/3}] \textstyle\sum_{p=0} K_p (2/x)^{(2/3)p}$$

where the summation is continued as long as the terms decrease. The K_p are evaluated most conveniently with a recursion formula where it is known that

TABLE 1

α_0	= 1.00000000	
α_1	= .886226926	
α_2	= −.500000000	
α_3	= −.295408975	
α_4	= .101459153	
α_5	= −2.95408975	$\times 10^{-2}$
α_6	= 5.78937917	$\times 10^{-3}$
α_7	= 4.68903135	$\times 10^{-4}$
α_8	= 6.4534565	$\times 10^{-5}$
α_9	= −2.6050174	$\times 10^{-6}$
α_{10}	= 2.7371871	$\times 10^{-7}$
α_{11}	= −6.7662790	$\times 10^{-9}$
α_{12}	= 5.7046675	$\times 10^{-10}$
α_{13}	= 9.6385741	$\times 10^{-12}$
β_0	= .0833333	
β_2	= 2.7777778	$\times 10^{-3}$
β_4	= 2.48015873	$\times 10^{-4}$
β_6	= 9.18577307	$\times 10^{-6}$
β_8	= 1.73972975	$\times 10^{-10}$
β_{10}	= 1.91179093	$\times 10^{-12}$

$K_0 = 1$. The formula given by Torrey[3] contains a misprint and should read:

$$36p\, K_p = (-12p^2 + 12p + 35) K_{p-1} - (1/6)\,(5 + 2p)\,(4 - 2p)\,(3 - 2p)\, K_{p-2}.$$

Using this formula the following values are obtained in agreement with those of Torrey:

$$K_0 = 1, \qquad K_1 = .972222, \qquad K_2 = .148534,$$

$$K_3 = -.017879, \qquad K_4 = .004594, \qquad K_5 = -.000762.$$

K_6 is already larger and is found to be −.001264. In this way, $I(x)$ and $K(x)$ are most easily computed by use of

$$I(x) \sim (\pi/3)^{1/2}(x/2) \exp [(-3/2)\,(x/2)^{2/3}]\,(A \sin\theta + B \cos\theta)$$

$$K(x) \sim -(\pi/3)^{1/2}(x/2) \exp [(-3/2)\,(x/2)^{2/3}]\,(A \cos\theta - B \sin\theta)$$

where $\theta = 3\sqrt{3}/2\ (x/2)^{2/3}$

$A = K_0 - K_3(2/x)^2 + (1/2)\ [K_1(2/x)^{2/3} - K_2(2/x)^{4/3} - K_4(2/x)^{8/3} + K_5(2/x)^{10/3}]$,

$B = (\sqrt{3}/2)[K_1(2/x)^{2/3} + K_2(2/x)^{4/3} - K_4(2/x)^{8/3} - K_5(2/x)^{10/3}]$

if terms only up to K_5 are included.

TABLE 2

x	$I(x)$	$K(x)$	x	$I(x)$	$K(x)$
0	.50000	0	6.4	−.0493	−.1535
.1	.49751	.04417	6.6	−.0307	−.1515
.2	.49019	.08754	6.8	−.0132	−.1476
.3	.47832	.12944	7.0	.00286	−.14211
.4	.46229	.16933	7.2	.01749	−.13518
.6	.41950	.24139	7.4	.03061	−.12709
.8	.36543	.30136	7.6	.04220	−.11805
1.0	.30366	.34805	7.8	.05224	−.10830
1.2	.23746	.38122	8.0	.06078	−.09808
1.4	.16972	.40127	8.5	.07562	−.07131
1.6	.10288	.40910	9.0	.08221	−.04496
1.8	.03892	.40592	9.5	.08191	−.02082
2.0	−.02062	.39314	10.0	.07626	−.00010
2.2	−.0746	.3722	10.5	.06684	.01654
2.4	−.1221	.3448	11.0	.05507	.02889
2.6	−.1629	.3122	11.5	.04224	.03707
2.8	−.1966	.2759	12.0	.02937	.04146
3.0	−.2233	.2371	12.5	.01727	.04259
3.2	−.2432	.1971	13.0	.00650	.04109
3.4	−.2565	.1569	13.5	−.00259	.03758
3.6	−.2639	.1173	14.0	−.00982	.03268
3.8	−.2657	.0792	14.5	−.01517	.02696
4.0	−.2626	.0430	15.0	−.01872	.02089
4.2	−.2552	.0094	15.5	−.02065	.01487
4.4	−.2441	−.0214	16.0	−.02118	.00921
4.6	−.2299	−.0490	16.5	−.02057	.00413
4.8	−.2132	−.0734	17.0	−.01906	−.00022
5.0	−.1945	−.0944	17.5	−.01691	−.00377
5.2	−.1745	−.1120	18.0	−.01435	−.00650
5.4	−.1536	−.1263	18.5	−.01159	−.00844
5.6	−.1322	−.1374	19.0	−.00879	−.00965
5.8	−.1108	−.1455	19.5	−.00609	−.01020
6.0	−.0896	−.1507	20.0	−.00360	−.01021
6.2	−.0691	−.1533			

In giving values for the asymptotic series we have assumed that the remainder is of the order of the last term used. Actually this appears to be a reasonable assumption since the value for $I(2)$ calculated from the asymptotic series differs only by .00005 from $I(2)$ as calculated from the convergent series. Both series were computed in the region $2 \leqq x \leqq 3$ and continued to give good agreement. For large x, the asymptotic series will approach the integral more closely. In this region the limit on accuracy was in the computation.

3. Numerical Results. A table of the functions $I(x)$ and $K(x)$ is given in Table 2.

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A NEW FORMULA FOR REPEATED MECHANICAL QUADRATURES

By Zdeněk Kopal, Pierre Carrus, and Katherine E. Kavanagh

The aim of the present note is to prove the existence of a quadrature formula of the form

$$\int_{-1}^{1}\left\{\int_{a_1}^{x}\phi(x)\,dx\right\}dx=\sum_{j=2}^{n}H_j^{(n)}\phi(a_j) \tag{1}$$

by means of which the definite integral on the left-hand side can be evaluated exactly—without remainder term—as a weighted mean of $n-1$ particular values of $\phi(x)$, provided that the degree of this function does not exceed $2n-1$. In order to attain this accuracy, the abscissae a_j and the corresponding weights $H_j^{(n)}$ must be determined in accordance with certain algebraic equations, which will be given below and solved numerically for $n=2, 3, 4, 5$, and 7.

In order to prove (1), consider the Hermite interpolation formula[1]

$$f(x)=\sum_{j=1}^{n}h_j(x)f(a_j)+\sum_{j=1}^{n}\bar{h}_j(x)f'(a_j)+\frac{\{p_n(x)\}^2}{(2n)!}f^{(2n)}(\xi) \tag{2}$$

of n-th degree, where ξ is a number intermediate between a_1 and a_n, and where we have abbreviated

$$h_j(x)=\{1-(x-a_j)[p_n''(a_j)/p_n'(a_j)]\}\,\{l_j(x)\}^2, \tag{3}$$

$$\bar{h}_j(x)=(x-a_j)\{l_j(x)\}^2 \quad (4), \qquad l_j(x)=\frac{p_n(x)}{(x-a_j)p_n'(a_j)}, \tag{5}$$

and

$$p_n(x)=(x-a_1)(x-a_2)\cdots(x-a_n) \tag{6}$$

respectively.

Integrating both sides of (2) between finite limits (which, without loss of generality, can be normalized to ±1) we obtain

$$\int_{-1}^{1}f(x)\,dx=\sum_{j=1}^{n}\left\{\int_{-1}^{1}h_j(x)\,dx\right\}f(a_j)+\sum_{j=1}^{n}\left\{\int_{-1}^{1}\bar{h}_j(x)\,dx\right\}f'(a_j)+R, \tag{7}$$

where

$$R=[1/(2n)!]\int_{-1}^{1}\{p_n(x)\}^2f^{(2n)}(\xi)\,dx.$$

If we now specify n arbitrary abscissae a_j by imposing that

$$\int_{-1}^{1}\bar{h}_j(x)\,dx=0 \qquad \text{for} \qquad j=1, 2, 3, \cdots, n, \tag{8}$$

[1] Cf., Szego, *Orthogonal Polynomials*, New York, 1939, pp. 323 f.

equation (7) reduces to the well-known Gauss quadrature formula,[3] with (8) rendering $p_n(x)$ orthogonal polynomials which (in the case of unit weight function and limits ± 1) reduce to the polynomial solutions of Legendre's equation for integral values of n. The respective quadrature formula will evidently permit us to evaluate exactly the definite integral on the left-hand side of (7) as a summation of n terms on the right-hand side provided that the degree of $f(x)$ is not in excess of $2n - 1$.

Suppose now, for the sake of argument, that we wish to derive from (7) an alternative quadrature formula by imposing that

$$\int_{-1}^{1} h_j(x)\,dx = 0 \qquad \text{for} \qquad j = 1, 2, 3, \cdots, n, \tag{9}$$

in place of (8). If so, the summation on the right-hand side of (7) will likewise reduce to n terms; but the abscissae a_j defined as roots of a simultaneous system of n non-linear equations of the form (9) prove to be complex. This fact could have been anticipated; for their reality would imply that the value of the definite integral of a function could be expressed as a summation of discrete values of its derivatives alone—an obviously impossible task.

Suppose, however, that in order to obviate this difficulty we propose to define our set of the n abscissae a_j by n conditions requiring that

$$\int_{-1}^{1} h_j(x)\,dx = 0 \qquad \text{for} \qquad j = 1, \tag{10}$$

and

$$\int_{-1}^{1} h_j(x)\,dx = 0 \qquad \text{for} \qquad j = 2, 3, 4, \cdots, n. \tag{11}$$

If so, equation (7) could be written as

$$\int_{-1}^{1} f(x)\,dx = H_1^{(n)} f(a_1) + \sum_{j=2}^{n} H_j^{(n)} f'(a_j) + R, \tag{12}$$

where

$$H_1^{(n)} = \int_{-1}^{1} h_1(x)\,dx, \tag{13}$$

while, for $j > 1$,

$$H_j^{(n)} = \int_{-1}^{1} h_j(x)\,dx. \tag{14}$$

Equation (12), without remainder term, should be exact for all functions $f(x)$ of degree not in excess of $2n - 1$. In particular, its application to $f(x) = 1$ discloses that

$$H_1^{(n)} = 2, \tag{15}$$

[3] Cf., Fort, *Finite Differences*, Oxford, 1948, pp. 110–111.

which permits us to rewrite (12) as

$$\int_{-1}^{1} \{f(x) - f(a_1)\}\, dx = \sum_{j=2}^{n} H_j^{(n)} f'(a_j) + R. \tag{16}$$

In order to express this result in a more symmetrical form, let us introduce a new function $\phi(x)$, defined by the relation

$$f(x) - f(a_1) = \int_{a_1}^{x} \phi(x)\, dx. \tag{17}$$

If we insert (17) in (16) this latter equation reduces readily to (1)—quod erat demonstrandum.

In order that a quadrature formula of the form (1) should exist, the reality of all roots a_j of our non-linear system (10)–(11) for any given value of n is a necessary prerequisite. A general proof of the nature of such roots for unrestricted n is, however, beyond the scope of the present note. With regard to the first root a_1, it is easy to prove that its value is necessarily interior to ± 1. Equation (10) discloses that

$$a_1 = \int_{-1}^{1} x\{l_1(x)\}^2\, dx \Big/ \int_{-1}^{1} \{l_1(x)\}^2\, dx, \tag{18}$$

which renders a_1 the centroid of a mass-distribution of a positive function within $x = \pm 1$; and if the a_j's for $j \neq 1$ are real, a_1 will be necessarily so. Moreover, if n is an odd integer, $a_1 = 0$ represents one of the admissible roots.

In order to investigate the nature of the roots of the simultaneous system represented by equations (10) and (11) in particular cases, we have evaluated such roots for $n = 2, 3, 4, 5$, and 7 by numerical processes. We found that not only are the roots real, but that there exists even a multiplicity of real solutions in each case considered to a sufficient detail. Their numerical values will be given below; and once they have been determined, the weights $H_j^{(n)}$ can be evaluated by straightforward quadratures. In order to reduce this task to a convenient form, consider the function $\phi(x) \equiv l_j(x)$, defined by (5) so that $l_j(a_i) = 0$ for $i \neq j$, while $l_j(a_j) = 1$. An application of our quadrature formula (1) to this function immediately yields

$$H_j^{(n)} = \int_{-1}^{1} \left\{ \int_{a_1}^{x} l_j(x)\, dx \right\} dx \tag{19}$$

for $j = 2, 3, \cdots n$. Moreover, since $l_j(x)$ is a polynomial of $(n - 1)$ degree and our quadrature formula is of the degree of precision $2n - 1$, the remainder term in (1) will also drop out for $f(x) \equiv \{l_j(x)\}^2$, leaving us with

$$H_j^{(n)} = \int_{-1}^{1} \left\{ \int_{a_1}^{x} \{l_j(x)\}^2\, dx \right\} dx \tag{20}$$

as an alternate expression for the desired weight coefficients. Unlike the classical Gauss case, however, the positivity of the function behind the second integral sign does not entitle us to conclude that the weights of the quadrature formula

(1) are all positive. In point of fact, negative weight coefficients arise already in connection with $n = 2$, as can be seen in the following tabulation:

$n = 2$	$a_1 = \pm .3933198$	$H_1 = 2$
	$a_2 = \mp .2270833$	$H_2 = \mp .7866396$
$n = 3$	$a_1 = 0$	$H_1 = 2$
	$a_2 = .5477225$	$H_2 = .3042903$
	$a_3 = -.5477225$	$H_3 = -.3042903$
	or	
	$a_1 = \pm .5889711$	$H_1 = 2$
	$a_2 = \pm .2250452$	$H_2 = \mp .8445196$
	$a_3 = \mp .5293628$	$H_3 = \mp .3334225$
$n = 4$	$a_1 = \pm .2556542$	$H_1 = \pm 2$
	$a_2 = \mp .14211735$	$H_2 = \mp .50874580$
	$a_3 = \pm .70293590$	$H_3 = \pm .13810482$
	$a_4 = \mp .70090019$	$H_4 = \mp .14066743$
$n = 5$	$a_1 = 0$	$H_1 = 2$
	$a_2 = \pm .372146$	$H_2 = \pm .299869$
	$a_3 = \pm .792006$	$H_3 = \pm .0695342$
	$a_4 = \mp .792006$	$H_4 = \mp .0695342$
	$a_5 = \mp .372146$	$H_5 = \mp .299869$
$n = 7$	$a_1 = 0$	$H_1 = 2$
	$a_2 = .282090$	$H_2 = .262524$
	$a_3 = .626860$	$H_3 = .115852$
	$a_4 = .882531$	$H_4 = .022651$
	$a_5 = -.882531$	$H_5 = -.022651$
	$a_6 = -.626860$	$H_6 = -.115852$
	$a_7 = -.282090$	$H_7 = -.262524$

The fact that, for even values of n, a_1 assumes a certain irrational value renders our quadrature formula (1) a rather special case. For odd values of n however, $a_1 = 0$ represents always an admissible solution; and repeated definite integrals of the form corresponding to $a_1 = 0$ are frequently encountered in applied mathematics, usually as a result of partial integration. It is to the evaluation of such integrals that the quadrature formula (1) should be primarily applicable.

In order to illustrate the use of the preceding quadrature formula on a practical case, consider the differential equation

$$y'' = \frac{1 + 3x + 2x^2}{(1 + x^2)\sqrt{1 + x^2}} \cos \sqrt{1 + x^2} - \frac{x^2(1 + x)}{1 + x^2} \sin \sqrt{1 + x^2}, \quad (21)$$

subject to the boundary conditions

$$y'(0) = 0, \qquad y(-1) = 1,$$

and let it be required to evaluate $y(1)$. The closed solution of the foregoing boundary-value problem can be found to be

$$y(x) = (1 + x)\{\sin \sqrt{1 + x^2} - \sin (1)\} + 1, \quad (22)$$

which yields

$$y(1) = 2\{\sin \sqrt{2} - \sin (1)\} + 1 = 1.2925899,$$

correctly to seven decimal places. If, however, we wish to forego a formal integration of (21) and to evaluate $y(1)$ approximately by the quadrature formula (1) developed in this paper, we set

$$y(x) = \int_{-1}^{x}\int_{0}^{x} y'' \, dx \, dx$$

and, using five points, we find that

$$y_5(1) = \sum_{j=1}^{5} H_j^{(n)} y''(a_j) = 1.291922,$$

which is in error by .000668 or approximately .07%. An application of the seven-point formula leads, moreover, to

$$y_7(1) = 1.292575,$$

which errs by defect of only .000015 or .002% of the true value.

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THE TRANSFER FUNCTION OF AN R–C LADDER NETWORK*

By Aaron Fialkow and Irving Gerst

Introduction. In the last few decades there has arisen an extensive literature concerned with the characterization and synthesis of various network functions. However, with few exceptions, these results make free use of ideal transformers and coupled coils, and the question of what can or cannot be achieved without such elements is at present not clearly understood. Also, there is little information on the effect of requiring the existence of a common ground terminal—the so-called unbalanced networks. Both of these problems, aside from their intrinsic interest, bear on the engineering design of practical networks.

The present paper throws some light on the above questions, in so far as the transfer function of a four-terminal network is concerned, in that we discuss networks with two kinds of elements, mutual inductance being excluded; and in that most of our results concern certain unbalanced networks of common occurrence. We may confine our discussion of the transfer function to the *R–C* case since the *R–L* and *L–C* cases follow easily from this. One method of characterizing a rational network function such as the transfer function of a four-terminal network, which we write in the form $A(p) = K\Pi(p + \delta_i)/\Pi(p + \gamma_i)$, is to give (*a*) the location of the poles (*b*) the location of the zeros relative to the poles (*c*) the range of the multiplicative constant K as a function of the zeros and poles. It was our aim to consider only networks for which we could obtain precise results concerning *all three* of these properties or any equivalent properties. Thus far we have succeeded in doing this completely for the *L*-network and the symmetrical lattice, and almost completely for the general ladder network. This is the reason for the choice of these particular structures in our discussion. The determination of (*c*) is one of the novel results of the paper. For each of the structures considered, a design procedure follows from the proof of realizability. While this may be no simpler than cut-and-try methods, it has an advantage over them in that one knows in advance whether the synthesis will succeed and thereafter there are no wasted steps.

We will now give a brief informal description of the results of our investigation. Detailed and precise definitions and statements of theorems are given in the text. Certain necessary properties of the transfer function of the most general four terminal *R–C* passive network are well known. Thus the transfer function must be a rational function with real coefficients which is regular at infinity. All of its poles must be simple and negative real. The latter property is (*a*) and holds for all the networks under consideration. There remains the description of (*b*) and (*c*).

Let us start with the *L*-network. Here the δ_i must be real and non-negative

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and obey certain order relations with respect to the γ_j which are analogous to the order relations for the impedance function of a two-terminal R–C network. In the latter case the sequence of δ_i and γ_j arranged in ascending order of magnitude consists of (γ, δ) pairs with possibly an unpaired γ termination. The corresponding sequence for the L-network transfer function consists of (γ, δ) and (δ, γ) pairs with possibly an unpaired γ termination (Thm. 1). For a given set of δ_i and γ_j satisfying these requirements, the range of K is an interval starting from zero and going up to a value K_0, beyond which the equation $\Pi(p + \gamma_j) - K\Pi(p + \delta_i) = 0$ for the first time fails to have all its roots real and non-positive. It follows from this that K_0 is determined as the smallest of (at most) three values each of which may be described in terms of the δ_i and γ_j (Thm. 2).

In the case of the symmetrical R–C lattice network there is no restriction on the δ_i other than that they be roots of a real polynomial. The range of K is the interval $-K_0 \leq K \leq K_0$, where K_0 is characterized exactly as in the L-network case.

Coming now to the R–C ladder network, we first consider networks having a fixed number of stages. The δ_i of such a network, besides being real and non-negative, must have certain order relations with respect to the γ_j. To describe these, let us again consider the sequence of δ_i and γ_j of an L-network or 1-stage ladder network. If we select any δ in this sequence and "look back" toward the beginning of the sequence, we will "see" either the same number of δ's as γ's or one more γ than δ. Also the total number of γ_j is at least equal to the total number of δ_i and cannot exceed this latter number by more than one. Similarly, the sequence of δ_i and γ_j of an s-stage ladder network may be characterized thus: If we select any δ in this sequence and "look back," we will "see" up to $(s - 1)$ more δ's than γ's or up to s more γ's than δ's. The total number of γ's is at least equal to the total number of δ's and does not exceed this latter number by more than s (Thm. 3). Except for the case $s = 1$, the range of K is as yet undetermined for an s-stage ladder network. If, however, we consider only those networks for which the zeros and poles of the component[1] L-network transfer functions $A_j(p)$, $(j = 1, 2, \cdots, s)$ are preassigned, then we find that $0 < K \leq K_0$ where K_0 is the product of the maximum K's associated with each $A_j(p)$(Thm. 4).

If we place no restriction upon the number of stages of the ladder then the order relation condition disappears and the only requirement on the δ_i is that they be real and non-negative. As for the range of K in this case, we have thus far been able to determine that $0 < K \leq K_0$ where K_0 is one of several quantities which depend on the δ_i and γ_j. However, we cannot at present determine in every case which one of these quantities is actually K_0.

I. THE L-NETWORK **1. The zeros and poles of the transfer function.** Consider the L-network shown in Figure 1. Here Z_a and Z_b are arbitary impedances consisting of resistance and capacitance only. As is well known,[2] Z_a and Z_b are

[1] This term is defined in §4.

[2] For example see E. A. Guillemin, *"Communication Networks"* (1935), Vol. 2, pp. 211–213.

rational functions having the form

$$Z_a = \eta \frac{(p+\alpha_2)(p+\alpha_4)\cdots}{(p+\alpha_1)(p+\alpha_3)\cdots}, \tag{1.1}$$

$$Z_b = \lambda \frac{(p+\beta_2)(p+\beta_4)\cdots}{(p+\beta_1)(p+\beta_3)\cdots}, \tag{1.2}$$

where p is the complex frequency and where the following relations are satisfied[3]

$$0 < \eta, \quad 0 < \lambda, \quad 0 \leq \alpha_1 < \alpha_2 < \alpha_3 < \cdots, \quad 0 \leq \beta_1 < \beta_2 < \beta_3 < \cdots. \tag{1.3}$$

The α and β sequences each terminate in a member whose subscript may be either odd or even.

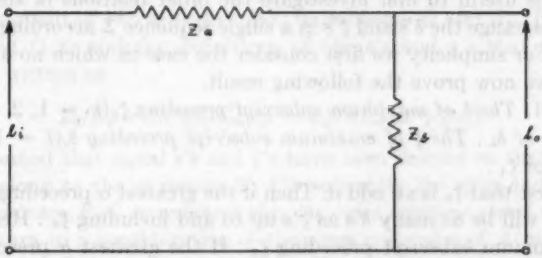

FIGURE 1

The *transfer function*, $A(p)$, of a four-terminal network is defined by $A(p) = e_0/e_i$, where e_0 and e_i are the output and input voltages respectively. In the case of the L-network, we readily find that

$$A(p) = Z_b/(Z_a + Z_b). \tag{1.4}$$

From equation (1.4) it immediately follows that the poles and zeros of $A(p)$ are all real and non-positive. For $Z_a + Z_b$ is again an R–C impedance and hence expressible in the form

$$Z_a + Z_b = \tau \frac{(p+\chi_2)(p+\chi_4)\cdots}{(p+\chi_1)(p+\chi_3)\cdots},$$

with

$$0 < \tau, \quad 0 \leq \chi_1 < \chi_2 < \chi_3 < \cdots.$$

From (1.1), (1.2) and (1.4) we find that

$$A(p) = \frac{\lambda(p+\beta_2)(p+\beta_4)\cdots(p+\alpha_1)(p+\alpha_3)\cdots}{\eta(p+\beta_1)(p+\beta_3)\cdots(p+\alpha_2)(p+\alpha_4) + \lambda(p+\beta_2)(p+\beta_4)\cdots(p+\alpha_1)(p+\alpha_3)\cdots}. \tag{1.5}$$

[3] The case where $\eta = 0$ or $\lambda = 0$ yields the trivial impedance functions $Z_a = 0$ or $Z_b = 0$ respectively.

For brevity we call an α whose subscript is odd or even an odd α or even α respectively, and use similar language relative to the β's. Arrange the even β's and odd α's in a single sequence according to ascending magnitude and designate them by $\delta_1, \delta_2, \cdots, \delta_N$. Similarly designate by $\zeta_1, \zeta_2, \cdots \zeta_P$ the odd β's and even α's arranged in a single sequence according to ascending magnitude.[4] Then (1.5) may be written as

$$A(p) = \frac{\lambda \prod_{i=1}^{N} (p + \delta_i)}{\eta \prod_{k=1}^{P} (p + \zeta_k) + \lambda \prod_{i=1}^{N} (p + \delta_i)} \tag{1.6}$$

From the definition of the δ's and ζ's it follows that either $P = N$ or $P = N \pm 1$.

In what follows we derive certain order relations concerning the zeros and poles of the transfer function $A(p)$. In this connection as well as for later application, it is useful to first investigate the order relations of the δ's and ζ's. To this end, arrange the δ's and ζ's in a single sequence Σ according to ascending magnitude. For simplicity we first consider the case in which no δ is equal to a ζ. About Σ we now prove the following result.

Lemma 1. *The δ of maximum subscript preceding $\zeta_k (k = 1, 2 \cdots, P)$ in Σ is either*[5] *δ_{k-1} or δ_k. The ζ of maximum subscript preceding $\delta_i (i = 1, 2, \cdots, N)$ is either ζ_{i-1} or ζ_i.*

Suppose first that ζ_k is an odd β. Then if the greatest α preceding ζ_k in Σ is an odd α, there will be as many δ's as ζ's up to and including ζ_k. Hence δ_k will be the δ of maximum subscript preceding ζ_k. If the greatest α preceding ζ_k is an even α, there will be one more ζ than δ up to and including ζ_k. Then δ_{k-1} will be the δ of maximum subscript preceding ζ_k.

On the other hand, let ζ_k be an even α. Then consider the β of maximum subscript which precedes it in Σ. By proceeding as above, we find that according as this maximum subscript is odd or even, the δ of maximum subscript preceding ζ_k is δ_{k-1} or δ_k respectively. This completes the proof of the first part of Lemma 1. The second sentence of the lemma is proved in precisely the same manner after interchanging the roles of δ and ζ and of α and β.

Lemma 1 is equivalent to the following Lemma which clearly exhibits the structure of the sequence Σ.

Lemma 2. *If the sequence Σ is written $\sigma_1, \sigma_2, \cdots, \sigma_{P+N}$, then each pair $(\sigma_{2k-1}, \sigma_{2k})$ consists of a δ and ζ which may appear in either order. If $P + N$ is odd then σ_{P+N} is unpaired and may be either a δ or a ζ.*

If $P + N < 2$, the result is trivial. If $P + N \geq 2$, suppose that the theorem is false. Then the first pair $(\sigma_{2k-1}, \sigma_{2k})$ not satisfying the theorem is either (ζ_k, ζ_{k+1}) or (δ_i, δ_{i+1}). In the first case, the δ of maximum subscript preceding both ζ_k and ζ_{k+1} is δ_k by the first part of Lemma 1. But by hypothesis there are exactly $k - 1$ pairs preceding ζ_k each containing one δ. Hence the δ of maximum sub-

[4] It is possible for at most two successive δ's to be equal, as when an odd α equals an even β. In this case it is immaterial in what order the α and β are arranged before designation as δ's with successive subscripts. Similar remarks apply to the ζ's.

[5] If $k = 1$ this statment is to be interpreted as meaning that either no δ or δ_1 precedes ζ_1. Similar remarks apply when $i = 1$.

script preceding ζ_k is δ_{k-1}. This contradiction eliminates the first possibility. The second possibility may be eliminated by using the second part of Lemma 1 in a similar manner. This completes the proof of Lemma 2.

We now consider the case in which some δ's and ζ's are equal. This may be regarded as a limiting case of that already discussed. It is obtained by letting some of the δ's and ζ's of the sequence Σ approach each other. It can be shown that a particular δ and ζ may become equal only if they belong to the same pair $(\sigma_{2k-1}, \sigma_{2k})$.

Form a new sequence Σ' by deleting all such equal δ's and ζ's from Σ. This will have the effect of removing the pair $(\sigma_{2k-1}, \sigma_{2k})$ if $\sigma_{2k-1} = \sigma_{2k}$. It is clear from this discussion that *the sequence Σ' has the same structural properties, described by Lemma 2, as the sequence Σ.*

Lemma 2 may be interpreted as describing the order relations of the zeros and poles of the quotient of two arbitrary R–C impedances Z_b and Z_a. For it follows from (1.1) and (1.2) and the definitions of the δ's and ζ's that the quotient Z_b/Z_a may be written as

$$Z_b/Z_a = (\lambda/\eta)[\Pi_{i=1}^{N}(p + \delta_i)/\Pi_{k=1}^{P}(p + \zeta_k)]. \tag{1.7}$$

Here it is assumed that equal δ's and ζ's have been deleted so that the δ's and ζ's of (1.7) belong to the sequence Σ'. Consequently *the zeros and poles of the ratio of two arbitrary R–C impedances obey the order relations stated in Lemma 2.*

We now return to a consideration of the zeros and poles of $A(p)$. Since (1.4) shows that $A(p)$ is the ratio of two R–C impedances, some information concerning its zeros and poles is given in the preceding paragraph. However, the special form of $A(p)$ imposes additional restrictions on its zeros and poles which we now discuss.

In the transfer function defined by equation (1.6), the numerator and denominator will have a common factor if and only if a δ and a ζ are equal. After dividing numerator and denominator by all common factors, suppose, for simplicity, that $A(p)$ is again written as (1.6). We note that the δ's and ζ's which now occur in (1.6) are the members of the sequence Σ', and that P and N now denote the number of ζ's and δ's respectively in Σ'. (But again $P = N$ or $P = N \pm 1$).

The zeros of the transfer function are $-\delta_1, -\delta_2, \cdots, -\delta_N$. It follows from the definition of the δ's that zeros of multiplicity at most two may occur except that $p = 0$ may be only a simple zero. The poles of $A(p)$ are the zeros of the algebraic equation

$$f(p) = 0 \tag{1.8}$$

where

$$f(p) = g(p) + h(p), \tag{1.9}$$

$$g(p) = \eta\Pi_{k=1}^{P}(p + \zeta_k), \qquad h(p) = \lambda\Pi_{i=1}^{N}(p + \delta_i).$$

According to Lemma 2 and the italicized statement which follows it, the k-th pair of Σ' is either $(\delta_k\zeta_k)$ or $(\zeta_k\delta_k)$.

In the first case,[6]

$$\operatorname{sgn} f(-\delta_k) = \operatorname{sgn} g(-\delta_k) = (-1)^{k-1}$$

$$\operatorname{sgn} f(-\zeta_k) = \operatorname{sgn} h(-\zeta_k) = (-1)^k$$

In the second case,

$$\operatorname{sgn} f(-\zeta_k) = \operatorname{sgn} h(-\zeta_k) = (-1)^{k-1}$$

$$\operatorname{sgn} f(-\delta_k) = \operatorname{sgn} g(-\delta_k) = (-1)^k.$$

In either case, there is at least one root of (1.8) between $-\delta_k$ and $-\zeta_k$. If $P + N$ is odd there is an unpaired final term in Σ' which may be either δ_{P+N} or ζ_{P+N}. In the first case,

$$\operatorname{sgn} f(-\delta_{P+N}) = \operatorname{sgn} g(-\delta_{P+N}) = (-1)^{P+N-1}$$

$$\operatorname{sgn} f(-\infty) = \operatorname{sgn} h(-\infty) = (-1)^{P+N}$$

In the second case,

$$\operatorname{sgn} f(-\zeta_{P+N}) = \operatorname{sgn} h(-\zeta_{P+N}) = (-1)^{P+N-1}$$

$$\operatorname{sgn} f(-\infty) = \operatorname{sgn} g(-\infty) = (-1)^{P+N}.$$

In these cases, there is at least one root of (1.8) between $-\delta_{P+N}$ and $-\infty$ or $-\zeta_{P+N}$ and $-\infty$ respectively.

Since the degree of $f(p)$ is equal to Max (P, N), we see that (1.8) will have exactly one root in each of the indicated intervals and that in this way, we get all the roots of (1.8). Consequently all the poles of $A(p)$ are negative real and simple. Let us denote these poles by $-\gamma_1, -\gamma_2, \cdots, -\gamma_M$ where $M = \text{Max } (P, N)$, and $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_M$. Arrange the δ's and γ's in a single sequence $S: s_1, s_2, \cdots, s_{M+N}$ according to ascending magnitude. The preceding remarks prove the following theorem concerning the structure of S.

Theorem 1. *Each pair* (s_{2k-1}, s_{2k}) *of* S *consists of a* δ *and a* γ *which may appear in either order. If* $M + N$ *is odd a final unpaired term of* S *will be present, in which case* $s_{M+N} = \gamma_M$.

This theorem establishes the order relations of the poles and zeros of $A(p)$. Thus, for example, the sequence $S_1: \delta_1, \gamma_1, \gamma_2, \delta_2, \delta_3, \gamma_3, \gamma_4$ may possibly be the sequence S associated with a transfer function $A(p)$ of an L-network, whereas the sequence $S_2: \delta_1, \gamma_1, \gamma_2, \gamma_3, \delta_2, \gamma_4, \delta_3$ is impossible for such a structure.

We note that Theorem 1 concerning the δ, γ sequence S is similar to Lemma 2 concerning the δ, ζ sequence Σ'. In fact, it is evident from the proof of Theorem 1 that we may combine both theorems into the following statement whose meaning will be clear from what has preceded. *The sequence formed by arranging the* δ*'s,* γ*'s and* ζ*'s in ascending order may be considered as a succession of triplets* (δ, γ, ζ) *or* (ζ, γ, δ) *except that possibly either one of the terminations* (δ, γ) *or* (ζ, γ) *may occur.*

[6] If a is real and $\neq 0$, $\operatorname{sgn} a = a/|a|$.

2. The realizability theorem. According to the results of §1, the transfer function of an R–C L-network is necessarily of the form

$$A(p) = K[\Pi_{i=1}^{N}(p + \delta_i)/\Pi_{j=1}^{M}(p + \gamma_j)] \tag{2.1}$$

where the γ_j are distinct positive numbers and the δ_i are non-negative real numbers,[7] no δ_i equals a γ_j, and the γ_j and δ_i together obey the order relations of Theorem 1. Here K is a positive constant and $M = N$ or $M = N + 1$. For brevity we shall refer to a function defined by (2.1) and having all the properties enumerated above as a *function of type L*. We shall now show that for a suitable range of values of K the converse is also true. More precisely, the following theorem is proved:

Theorem 2. *The necessary and sufficient condition that a non-constant*[8] *function $A(p)$ of type L be the transfer function of a passive L-network containing resistance and capacitance only is that $0 < K \leq K_0$, where K_0 is the least of the three quantities K_d, $\Pi_{j=1}^{M}\gamma_j/\Pi_{i=1}^{N}\delta_i$, 1 if $M = N$ and of the first two quantities if $M = N + 1$.*[9] *Here K_d is the least positive root (if it exists) of the equation in K obtained by equating the discriminant of $\Pi_{j=1}^{M}(x + \gamma_j) - K\Pi_{i=1}^{N}(x + \delta_i) = 0$, to zero.*

We require three preliminary lemmas.

Lemma 3. *Let $\delta_i (i = 1, 2, \cdots, N)$ and $\zeta_k (k = 1, 2, \cdots, P)$ be distinct non-negative real numbers which when arranged in ascending order form a sequence $\bar{\Sigma}$ which obeys the order relations of Lemma* 2. *Then except for arbitrary multiplicative factors η and λ, there exist exactly*[10] $2^{[(N+P)/2]}$ *distinct impedance pairs Z_a and Z_b as defined by* (1.1), (1.2) *and* (1.3) *whose sequence Σ coincides with the given sequence $\bar{\Sigma}$.*

The proof is by induction. We recall that the δ's must be assigned as odd α's or even β's and the ζ's as even α's or odd β's. If $\bar{\Sigma}$ is δ_1 or ζ_1 the result is evident. Otherwise if (δ_1, ζ_1) is the first pair[11] of $\bar{\Sigma}$, we may make the following identifications: $\delta_1 = \alpha_1$, $\zeta_1 = \alpha_2$ or $\delta_1 = \alpha_1$, $\zeta_1 = \beta_1$. Similarly, if (ζ_1, δ_1) is the first pair there are again two possible identifications. Suppose that the δ's and ζ's of the first k pairs in $\bar{\Sigma}$ have been identified as[12] $\alpha_1, \alpha_2, \cdots, \alpha_i$; $\beta_1, \beta_2, \cdots, \beta_j$. Then $i + j = 2k$. If there is a $(k + 1)$-th pair in $\bar{\Sigma}$ the identification may be continued in two ways. We illustrate for the case in which i is even and $(\delta_{k+1}, \zeta_{k+1})$ is the next pair. Take $\alpha_{i+1} = \delta_{k+1}$, $\alpha_{i+2} = \zeta_{k+1}$; or $\alpha_{i+1} = \delta_{k+1}$, $\beta_{j+1} = \zeta_{k+1}$. The latter is possible since j is also even. The other cases are treated similarly. If $\bar{\Sigma}$ terminates in a δ or a ζ after the k-th pair, the same procedure

[7] Throughout the paper, when such sequences of numbers occur, it is to be understood that the indices have been assigned so that when the numbers have been arranged according to their subscripts, their absolute values form an increasing monotonic sequence.

[8] If $A(p)$ is a constant K, then it readily follows that an L-network realizing $A(p)$ exists if and only if $0 \leq K \leq 1$. The equality signs obtain only when either Z_a or Z_b is a zero impedance function.

[9] If $\delta_i = 0$ the quantity $\Pi\gamma_j/\Pi\delta_i$ is to be interpreted as ∞.

[10] Here the bracket indicates the greatest integer symbol.

[11] This and subsequent pairs are those referred to in Lemma 2.

[12] If no α's have appeared thus far, take $i = 0$ in what follows. A similar remark applies to the β's and j.

shows there is just one way of completing the identification. Consequently, in all, the number of distinct determinations of sets α_i, β_j (each leading to a distinct pair Z_a and Z_b) will be 2^R, where R is the number of pairs occurring in Σ. The latter number is clearly $[(N + P)/2]$. This completes the proof of Lemma 3.

The modifications which must be made in Lemma 3 when equal δ's or equal ζ's are allowed in Σ are considered next. Suppose $\delta_k = \delta_{k+1}$. Then if $\delta_k = \alpha_i$, $\delta_{k+1} = \beta_j$, (i odd, j even) is one identification of these δ's, the identification $\delta_k = \beta_j$, $\delta_{k+1} = \alpha_i$ is another permissible one which, however, leads to the same α, β sets as the first. Thus the number of distinct impedances Z_a and Z_b is halved. The same will apply for each equal pair of δ's or ζ's. Summing up we have the

Lemma 4. *If the hypothesis of Lemma* 3 *is modified so that* Σ *contains* κ *pairs of equal* δ*'s and* ρ *pairs of equal* ζ*'s, then the number of distinct impedance pairs* Z_a, Z_b *is given by* $2^{[(N+P)/2]-(\kappa+\rho)}$.

In any case, there will be at least one pair of impedances Z_a, Z_b corresponding to Σ.

Just this last fact follows immediately by taking

$$Z_a = \eta(p + \zeta_2)(p + \zeta_4) \cdots/(p + \delta_1)(p + \delta_3) \cdots,$$
$$Z_b = \lambda(p + \delta_2)(p + \delta_4) \cdots/(p + \zeta_1)(p + \zeta_3) \cdots.$$

The complete results of Lemmas 3 and 4 are required in connection with the more general synthesis procedure of the next section.

Lemma 5. *Let* $F(p, K) = \Pi_{j=1}^{M}(p + \gamma_j) - K\Pi_{i=1}^{N}(p + \delta_i)$ *where* γ_j, δ_i *are any complex numbers and* $\gamma_j \neq \delta_i$. *If* $K_1 \neq K_2$ *are any two complex numbers, then the equations* $F(p, K_1) = 0$ *and* $F(p, K_2) = 0$ *cannot have a common root.*

For if $p = -\zeta$ were such a root, then subtraction of the equations would yield $(K_1 - K_2)\Pi_{i=1}^{N}(-\zeta + \delta_1) = 0$. Consequently $\zeta = \delta_i$ for some i and using $F(-\zeta, K_1) = 0$ we would find $\delta_i = \gamma_j$ for some j, contrary to hypothesis. Since the roots of an algebraic equation are continuous functions of the coefficients, this result implies that if, as K ranges over a real interval, the roots of $F(p, K) = 0$ are real, then these roots are strictly monotonic functions of K.

We return now to the proof of Theorem 2, considering first the sufficiency of the conditions. In order to exhibit two impedance functions Z_a, Z_b which lead to the given transfer function $A(p)$, it suffices to show that for each K of the interval $0 < K \leq K_0$ we have identically

$$\Pi_{j=1}^{M}(p + \gamma_j) - K\Pi_{i=1}^{N}(p + \delta_i) \equiv K(\eta/\lambda)\Pi_{k=1}^{P}(p + \zeta_k), \tag{2.2}$$

where the $\zeta_k \neq \delta_i$ are non-negative real numbers, which together with the δ_i form a sequence obeying the order relations of Lemma 2, and $\eta > 0$, $\lambda > 0$. For then by Lemma 4, a Z_a and Z_b having multiplicative factors η and λ respectively exist whose sequence Σ coincides with the sequence of δ's and ζ's. Consequently using (2.2) in (1.6) which is the expression for the transfer function corresponding to Z_a and Z_b we find that (2.1) results.

Again denoting the left-member of (2.2) by $F(p, K)$, the ζ_k are the negatives of the roots of $F(p, K) = 0$. The roots of the equation $F(p, 0) = 0$ are the

$-\gamma_j$, $(j = 1, 2, \cdots, M)$. Since $A(p)$ is of type L, the γ_j themselves have the properties required by the ζ_k in (2.2). It follows from the continuity of the roots of an algebraic equation, in this case considered as functions of K, that for K sufficiently small and positive say $0 < K \leq K_\epsilon < 1$ the ζ_k will be close to the γ_j. Consequently for these values of K, the ζ_k will be distinct positive numbers different from the δ_i and together with these forming a sequence obeying the order relations of Lemma 2. Also by comparing coefficients of the highest powers of p in both members of (2.2) we find since $K_\epsilon < 1$ that $\eta/\lambda = 1/K$ or $1/K - 1$ according as $M = N + 1$ or $M = N$. Thus, η/λ will be positive and hence η and λ may both be chosen as positive numbers. In (2.2), by transposing the second product on the left, the right member takes the same form as $f(p)$ in (1.9). Hence the location of the γ_j is determined from that of the δ's and ζ's as in the proof of Theorem 1 and the italicized statement following that theorem holds for the present (ζ, γ, δ) sequence. However here only the termination (ζ, γ) is possible since we have the same number of ζ's as γ's. Furthermore in view of Lemma 5, the ζ_k will vary in a monotonic manner from their initial values γ_j as K varies from 0 to K_ϵ. .

Now let K increase positively, starting from K_ϵ. Then the ζ_k will vary continuously with K. No ζ_k can become equal to a δ_i or a γ_j as (2.2) would then imply that a δ_i and γ_j were equal. Hence as long as the ζ_k remain non-negative real numbers they will maintain their relative order with respect to the δ_i and γ_j. In order for the ζ_k to lose their non-negative real character one of the following must occur:

(a) a ζ_k becomes zero.

(b) a ζ_k becomes infinite.

(c) Two of the ζ_k become equal.

At least one of (a), (b), (c) will occur for some finite K. Otherwise the initial order relation of the (ζ, γ, δ) sequence for small K would be maintained for arbitrary large values of K. But as $K \to \infty$, (2.2) shows that the ζ_k will approach the δ_i if $M = N$; and if $M = N + 1$ all but one ζ_k will approach the δ_i the remaining ζ_k becoming infinite. A consideration of the termination of the (ζ, γ, δ) sequence in each case shows that this is impossible if the ζ_k remain non-negative real. Let K_0 denote the smallest positive value of K for which at least one of (a), (b), (c) obtains. Then evidently for $0 < K \leq K_0$ the ζ_k (if (b) occurs, the remaining ζ_k) will be non-negative and have the required order relations with respect to the δ_i.

Case (a) will occur if $\delta_1 \neq 0$ and $K = \Pi_{j=1}^{M}\gamma_j/\Pi_{i=1}^{N}\delta_i$. For small K, the (ζ, γ, δ) sequence is either (ζ_1, γ_1) or has an initial triple of the form $(\zeta_1, \gamma_1, \delta_1)$. Case (b) can occur only when both $M = N$ and $K = 1$. For small K the (ζ, γ, δ) sequence terminates in the triple $(\delta_N, \gamma_N, \zeta_N)$. Case (c) will occur for values of K which make the discriminant of $F(p, K) = 0$ vanish. For small K the (ζ, γ, δ) sequence contains at least two successive triplets of the form $(\delta_i, \gamma_i, \delta_i)$, $(\zeta_{i+1}, \gamma_{i+1}, \delta_{i+1})$ or it terminates in $(\delta_{M-1}, \gamma_{M-1}, \zeta_{M-1})$, (ζ_M, γ_M). The value of K_0 given in the statement of Theorem 2 follows from these results.

There remains the verification of the fact that η and λ may be chosen as positive numbers for $0 < K \leq K_0$. By equating coefficients of the highest powers of p in (2.2), we find

$$(2.3)\qquad \begin{cases} \eta/\lambda = 1/K, & M = N+1 \\ \eta/\lambda = (1-K)/K, & M = N, K < 1 \\ \eta/\lambda = \Sigma_{i=1}^{M}(\gamma_i - \delta_i), & M = N, K = 1. \end{cases}$$

The ratio η/λ which is determined by either of the first two equations is clearly positive. To establish the sign of the expression $\Sigma_{i=1}^{M}(\gamma_i - \delta_i)$ which occurs in the third equation, consider the coefficient of p^{M-1} in both members of (2.2) when $K < 1$. In this way, using the second equation of (2.3), we obtain the equation

$$\Sigma_{i=1}^{M}(\gamma_i - K\delta_i) = (1-K)\Sigma_{k=1}^{M}\zeta_k$$

from which it follows that $\Sigma_{i=1}^{M}(\gamma_i - \delta_i) = \lim_{K\to 1}\Sigma_{i=1}^{M}(\gamma_i - K\delta_i) \geq 0$. If the equality sign were to hold, then the coefficients of p^M and p^{M-1} of $F(p, K)$ would be zero for $K = 1$. Reference to (2.2) shows that at least two ζ's have become infinite for $K = 1$ which is impossible in view of the order relations of the (ζ, γ, δ) sequence. Therefore, in every instance, η and λ may both be chosen as positive numbers. This completes the proof of the sufficiency statement in Theorem 2.

We now consider the necessity of the condition stated in the theorem. If Z_a and Z_b realizing $A(p)$ are given, then a comparison of (1.6) and (2.1) shows that (2.2) must hold. Furthermore, the order relations stated after Theorem 1 for the (ζ, γ, δ) sequence are obeyed. Here the δ_i, ζ_k refer to the sequence Σ' associated with Z_a and Z_b. Now as K varies from 0 to K_0, the roots ζ_k of $F(p, K) = 0$ will completely cover one of the closed intervals: $[0, \gamma_1]$ for Case (a), $[\gamma_M, \infty]$ for Case (b) or $[\gamma_j, \gamma_{j+1}]$ for at least one $j(1 \leq j < M)$ in Case (c). Hence, by Lemma 5, for no value of $K > K_0$ can a root of $F(p, K) = 0$ belong to one of the covered intervals. Therefore, if $K > K_0$ it is impossible for the relation (2.2) to hold with the proper order relations for the (ζ, γ, δ) sequence. This completes the proof of Theorem 2.

By applying a similar method in the case of an R–C symmetric lattice, we arrive at the following result: *Necessary and sufficient conditions for the rational function*

$$A(p) = K\Pi_{i=1}^{N}(p + \delta_i)/\Pi_{j=1}^{M}(p + \gamma_j)$$

to be the transfer function of a symmetric lattice are

(i) The γ_j are distinct positive numbers.

(ii) The δ_i are any real numbers or occur in conjugate complex pairs.

(iii) $N \leq M$.

(iv) K is real and such that $0 < |K| \leq K_0$ where K_0 is the smallest of the three quantities $|K_d|$, $\Pi_{j=1}^{M}\gamma_j/\Pi_{i=1}^{N}|\delta_i|$, 1 if $N = M$ and of the first two quantities if

$N < M$. Here K_d is the real root of smallest absolute value (if it exists) of the equation in K obtained by equating the discriminant of

$$\Pi_{j=1}^{M}(x + \gamma_j) - K\Pi_{i=1}^{N}(x + \delta_i) = 0,$$

to zero.

To conclude this section we will sketch an alternative method for obtaining the maximum allowable value of K for the L-network which is more in line with standard network-theory practice.[13]

In the notation of Guillemin,[14] if Z_{11} and Z_{21} denote open-circuit driving point and transfer impedances respectively of a four-terminal network, we have

$$A(p) = Z_{21}/Z_{11}.$$

Given the L-function

$$A(p) = K\Pi_{i=1}^{N}(p + \delta_i)/\Pi_{j=1}^{M}(p + \gamma_j),$$

we will determine for what values of K functions Z_{11} and Z_{12} exist so as to be open-circuit driving point and transfer impedances respectively of an L-network. For this type of network we readily find from Figure 1 that

$$Z_{21} = Z_b, \qquad Z_{11} = Z_a + Z_b.$$

Hence both Z_{11} and Z_{21} are R–C impedance functions. Since an R–C impedance has a positive residue at a pole, it follows that every pole of Z_{21} must be a pole of Z_{11}, and that the residues at the poles of Z_{21} cannot exceed the residues at the corresponding poles of Z_{11}. This latter condition is simply the residue requirement for a four-terminal network as applied to the L-network.[15] We note that the functions Z_a and Z_b are immediately determinable once Z_{11} and Z_{21} are known and hence the corresponding network can be synthesized.

We may write

$$Z_{21} = K_2\Pi(p + \delta_i)/(p + \sigma_k), \qquad K_2 > 0$$

$$Z_{11} = K_1\Pi(p + \gamma_j)/(p + \sigma_k), \qquad K_1 > 0,$$

where $K = K_2/K_1$ and where in order to get R–C impedances the σ_k must be distinct, non-negative numbers such that

$$0 \leq \sigma_1 \leq \delta_1 \leq \sigma_2 \leq \delta_2 \cdots,$$

$$0 \leq \sigma_1 < \gamma_1 < \sigma_2 < \gamma_2 \cdots.$$

Thus the σ_k must lie in the intervals

$$0 \leq \sigma_1 \leq \text{Min}\,(\delta_1, \gamma_1),$$

$$\text{Max}\,(\delta_1, \gamma_1) \leq \sigma_2 \leq \text{Min}\,(\delta_2, \gamma_2),$$

. . . . . .

[13] We are indebted to Prof. R. M. Foster of the Polytechnic Institute of Brooklyn for reading the manuscript and suggesting this approach.

[14] E. A. Guillemin, loc. cit., p. 132–136.

[15] E. A. Guillemin, loc. cit., p. 217.

where the equality sign may hold only between a σ and a δ. These intervals for σ_i correspond to the exterior of the intervals determined by the zero-pole pairs of $A(p)$ except possibly for the last pair. For definiteness consider the case where $N = M$ and we have N σ's. The other possibilities may be treated in a similar manner. For any choice of the σ_k in the above intervals, Z_{21} and Z_{11} will be R–C impedance functions. Writing these functions in their canonical partial fraction expansion form, we have

$$Z_{21} = K_2\{1 + \Sigma[A_i/(p + \sigma_i)]\}$$

$$Z_{11} = K_1\{1 + \Sigma[B_i/(p + \sigma_i)]\}$$

where $A_i \geq 0$ and $B_i > 0$.

Since $Z_a(\infty) \geq 0$,

$$K_1 \geq K_2 .$$

From the residue condition there results

$$B_iK_1 \geq A_iK_2 .$$

With the notation $f(p) = \Pi(p + \gamma_j)$, $g(p) = \Pi(p + \sigma_k)$, $h(p) = \Pi(p + \delta_i)$, the A_i and B_i are given by

$$A_i = h(-\sigma_i)/g'(-\sigma_i), \qquad B_i = f(-\sigma_i)/g'(-\sigma_i),$$

and hence the inequalities may be rewritten as

$$K \leq 1, \qquad K \leq f(-\sigma_i)/h(-\sigma_i).$$

As σ_i ranges over the closed interval $\text{Max}\ (\delta_{i-1}, \gamma_{i-1}) \leq \sigma_i \leq \text{Min}\ (\delta_i, \gamma_i)$, let m_i denote the maximum of $f(-\sigma_i)/h(-\sigma_i)$ in this interval, allowing ∞ as a possible value of m_i. Then the maximum value K^* of K leading to functions Z_{11} and Z_{21} for this case is given by

$$K^* = \text{Min}\ [1, m_1, m_2, \cdots, m_i, \cdots m_N].$$

If $N = M$ we may also have $N + 1$ σ's and for this case we find,

$$K^* = \text{Min}\ [m_1, m_2, \cdots, m_i, \cdots, m_N, m_{N+1}],$$

where m_{N+1} is determined for the interval $\text{Max}\ (\delta_N, \gamma_M) \leq \sigma_{N+1}$. If $M = N + 1$ we can have only $N + 1$ σ's and then

$$K^* = \text{Min}\ [m_1, m_2, \cdots, m_i, \cdots, m_{N+1}]$$

where m_{N+1} is determined for the interval $\text{Max}\ (\delta_N, \gamma_N) \leq \sigma_{N+1} \leq \gamma_M$.

The actual analytical evaluation of the m_i which appear in these formulas leads to a consideration of the roots of the discriminant equation and, in certain circumstances, of the ratio $\Pi\gamma_j/\Pi\delta_i$ and 1. However, the result thus obtained is not identical with that stated in Theorem 2 and, indeed, is considerably weaker. What is lacking in the present derivation of the formulas for K^* and is needed to bridge the gap between the two results are the continuity properties of the roots of an algebraic equation.

3. The synthesis procedure. We now consider the question of synthesizing an L-network from a preassigned transfer function $A(p)$ of type L. We shall suppose that $A(p)$ is given by (2.1) where the γ's and δ's are known quantities but where K is as yet undetermined. Then the results of §2 yield the following algorithm for performing the required synthesis:

(a) Determine K_0 as the smallest of the three quantities

$$K_d\,,\ \Pi_{j=1}^{M}\gamma_j/\Pi_{i=1}^{N}\delta_i\,,\ 1 \tag{3.1}$$

if $M = N$ and of the first two quantities if $M = N + 1$.

We recall that K_d was defined as the least positive root of the equation in K obtained by setting the discriminant[16] of

$$\Pi_{j=1}^{M}(x+\gamma_j) - K\Pi_{i=1}^{N}(x+\delta_i) = 0 \tag{3.2}$$

equal to zero. In view of the difficulty of calculating K_d and in order to avoid unnecessary computation some remarks relative to the determination of K_0 are in order. Let S, as before, denote the sequence of γ_j, δ_i arranged in order of increasing numerical magnitude. For certain sequences S the following rules which follow readily from the proof of Theorem 2 simplify the determination of K_0:

(i) If no γ's occur successively in S, K_d need not be calculated.

(ii) If S begins with a δ, $\Pi_{j=1}^{M}\gamma_j/\Pi_{i=1}^{N}\delta_i$ may be omitted from consideration.

(iii) If S terminates in a δ, the number 1 may be omitted in (3.1).

(iv) If S begins with a δ and terminates in either a δ or two consecutive γ's then $K_0 = K_d$.

For sequences S to which none of the above rules apply it can be shown by examples that any one of the three quantities in (3.1) may actually be equal to K_0.

(b) Choose a value of K such that $0 < K \leq K_0$.

(c) For this value of K, solve (3.2) and denote its roots by $-\zeta_1$, $-\zeta_2$, $\cdots -\zeta_P$, where $P = M$ or $P = M - 1$. The ζ_k will all be real and non-negative, and may include double roots. In solving (3.2) it is of aid to first locate the ζ_k by noting that the (ζ, γ, δ) sequence must divide into triplets (ζ, γ, δ) or (δ, γ, ζ) except for possible terminations (δ, γ) and (ζ, γ).

(d) The α's and β's which describe the zeros and poles of Z_a and Z_b may now be determined as follows: Arrange all the δ's and ζ's in a single sequence Σ' according to ascending numerical order. Beginning with the smallest quantity in this sequence and proceeding to each successive quantity in turn each δ and ζ will now be identified as an α or β according to the following rules:

(i) If Σ' is δ_1 let $\alpha_1 = \delta_1$; if Σ' is ζ_1 let $\beta_1 = \zeta_1$. Otherwise the sequence Σ' of δ's and β's may begin as either δ_1, ζ_1 or ζ_1, δ_1. In the first case identify δ_1 as α_1 and ζ_1 as either α_2 or β_1. In the second case, identify ζ_1 as β_1 and δ_1 as either α_1 or β_2.

[16] For the expression of the discriminant of an algebraic equation in terms of its coefficients see e.g. L. E. Dickson, "*Elementary Theory of Equations*" (1922), pp. 146, 152.

(ii) The process of identification may be continued as follows: Suppose at some stage a certain number of δ's and ζ's of Σ' have been identified as[17]: α_1, α_2, $\cdots$, α_{i-1}; β_1, β_2, $\cdots$, β_{j-1} in some order. Then if the next quantity in the sequence Σ' is a δ it may be identified as either α_i if i is odd, or β_j if j is even. If the next quantity in the sequence Σ' is a ζ however, it may be identified as either α_i if i is even, or β_j if j is odd.

In making these identifications, the decision as to whether to identify a given quantity in the sequence as α_i or β_j (when both identifications are possible) is entirely arbitrary. As shown in Lemmas 3 and 4, there will be in all $2^{[(N+P)/2]-(\alpha+\rho)}$ ways of determining the α's and β's.

Since no δ is equal to a ζ in Σ' the foregoing process will yield a Z_a and Z_b which have no zeros or poles in common. To get the most general α and β sequences associated with $A(p)$, any number of distinct pairs $\bar{\delta}_i = \bar{\zeta}_k$, where the $\bar{\delta}_i$, $\bar{\zeta}_k$ are non-negative and different from the δ_i and ζ_k may be added to form a new sequence Σ. The identification procedure is then applied to Σ. However, since the resulting Z_a and Z_b will be of greater complexity (higher degree) than the Z_a and Z_b arising from Σ', we shall restrict ourselves only to this latter case for the present.[18]

(e) The quantities η and λ which fix the impedance levels of Z_a and Z_b may be determined from (2.3). It is clear that (2.3) only fixes the ratio η/λ of the impedance levels. Thus the value of either η or λ, but not both, may be chosen arbitrarily.

(f) By the preceding steps, Z_a and Z_b have been determined as rational expressions of the form (1.1), (1.2) for which (1.3) holds. The network synthesis of these impedances is carried through according to the classical methods of Foster or Cauer.[19] If desirable the resulting canonical impedances may then be transformed into any equivalent impedances in the customary manner using matrix algebra.[20]

The preceding synthesis procedure may be modified to accomodate any given resistive source and load. In the general case, this may require a finite reduction in the value of the maximum realizable K. However, if the source is zero or the load is infinite, all K except possibly K_0 may be realized.

The least number of elements required for the realization of $A(p)$ as an L-network is easily given. For it is known that the least number of elements by which an R-C impedance may be realized is one more than the total number of its internal zeros and poles. By an internal zero or pole is meant one not equal to zero or infinity. Now the finite zeros and poles of Z_a and Z_b in their totality coincide with the δ's and ζ's. Hence we require $N' + P' + 2$ elements to realize

[17] If no α's have been identified, apply rule (ii), using α_1 for α_i. Similarly if no β's have been identified, apply rule (ii) using β_1 for β_j.

[18] But note that the addition of the pair (0, 0) when possible will not increase the number of elements required for the realization of $A(p)$ as will be seen later.

[19] For example, see E. A. Guillemin, loc. cit., p. 211–213.

[20] E. A. Guillemin, loc. cit., p. 229–254.

$A(p)$, where N' and P' denote the number of non-zero δ's and ζ's associated with $A(p)$.

This number may also be expressed in terms of the number of zeros and poles of $A(p)$. We have $P' = P$ or $P' = P - 1$. The latter case obtains when $\zeta_1 = 0$ and this will occur if and only if $K = \Pi_{j=1}^{M}\gamma_j/\Pi_{i=1}^{N}\delta_i$. If we let K' denote the D.C. response $A(0)$ then from (2.1),

$$K' = K\Pi_{i=1}^{N}\delta_i/\Pi_{j=1}^{M}\gamma_j . \tag{3.3}$$

Hence $\zeta_1 = 0$ if and only if $K' = 1$. Also $P = M$ or $M - 1$ depending on whether $K \neq 1$ or $K = 1$. Consequently the least number of elements is given by

$$\begin{aligned} &N' + M + 2 \quad \text{if} \quad K' < 1, K \neq 1 \\ &N' + M + 1 \quad \text{if} \begin{cases} K' = 1, K \neq 1 \\ K' < 1, K = 1 \end{cases} \\ &N' + M \qquad \text{if} \quad K' = 1, K = 1. \end{aligned}$$

We now illustrate the synthesis method just explained by an example. The successive steps are lettered to correspond with those of the algorithm. *Example*:

$$A(p) = K(p + 5)(p + 6)/(p + 1)(p + 50).$$

(a) For this function, $\delta_1 = 5$, $\delta_2 = 6$, $\gamma_1 = 1$, $\gamma_2 = 50$, so that the sequence S is γ_1, δ_1, δ_2, γ_2. By (*i*) K_d need not be calculated. Hence

$$K_0 = \text{Min}\,[1, 1 \times 50/5 \times 6] = 1.$$

(b) Any value of K such that $0 < K \leq 1$ will yield a realizable function $A(p)$. Let us choose $K = 1$.

(c) Then equation (3.2) becomes

$$(x + 1)(x + 50) - (x + 5)(x + 6) = 0.$$

Solving, we find $\zeta_1 = 0.5$.

(d) Hence the sequence Σ' is: $\zeta_1 = 0.5$, $\delta_1 = 5$, $\delta_2 = 6$. Identification of the members of Σ' according to rules (*i*) and (*ii*) leads to the sets

$$(1) \qquad \alpha_1 = 5; \quad \beta_1 = 0.5, \quad \beta_2 = 6;$$

$$(2) \qquad \alpha_1 = 6; \quad \beta_1 = 0.5, \quad \beta_2 = 5.$$

The number of sets is in accord with the formula $2^{[(N+P)/2]-(k+\rho)}$ for here $N = 2$, $P = 1$, $\kappa = 0$, $\rho = 0$.

(e) According to (2.3), ($M = N = 2$, $K = 1$).

$$\eta/\lambda = (1 + 50) - (5 + 6) = 40$$

so that $\eta = 40\,\lambda$.

(f) We have now determined the following expressions for Z_a and Z_b, in which the multiplicative factors are all expressed in terms of λ.

(1) $Z_a = 40\lambda/p + 5), \quad Z_b = \lambda(p + 6)/(p + .5)$

(2) $Z_a = 40\lambda/(p + 6), \quad Z_b = \lambda(p + 5)/(p + .5).$

The figure illustrates the Foster realization of the first of these.

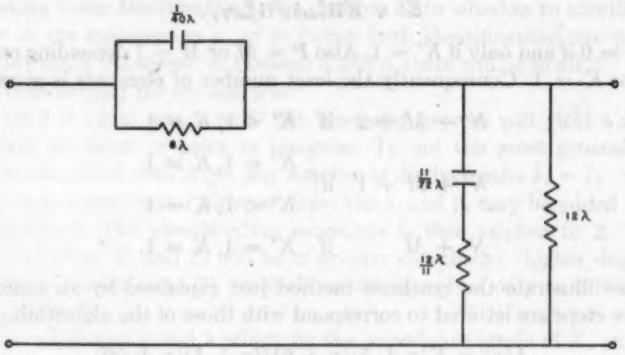

FIGURE 2

II. THE LADDER NETWORK.[21] **4. The zeros and poles of $A(p)$.** Consider the s-stage ladder network shown in Figure 3.

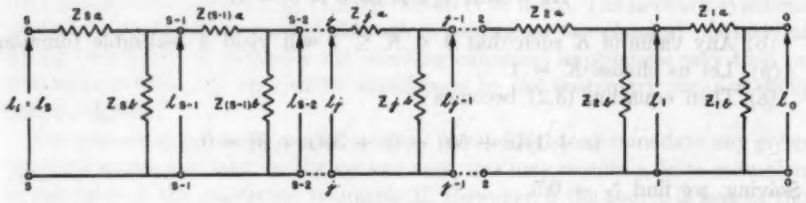

FIGURE 3

The numbered terminal pairs in Figure 3 divide the network into stages. Thus we shall call the section between terminal pairs 0 and 1 the first stage, between 1 and 2 the second stage, etc. Here the Z_{ja} and Z_{jb}, $(j = 1, 2, \cdots, s)$ are arbitrary R-C impedances which may be written in the form

$$Z_{ja} = \eta_j \frac{P_j}{Q_j} = \eta_j \frac{(p + \alpha_{j2})(p + \alpha_{j4}) \cdots}{(p + \alpha_{j1})(p + \alpha_{j3}) \cdots},$$

(4.1)

$$Z_{jb} = \lambda_j \frac{R_j}{T_j} = \lambda_j \frac{(p + \beta_{j2})(p + \beta_{j4}) \cdots}{(p + \beta_{j1})(p + \beta_{j3}) \cdots},$$

The n_j, λ_j, α_{jk} and β_{jk} satisfy relations of the type (1.3).

[21] The results on the ladder network are obtained essentially by means of an induction starting with the L-network. We will therefore in Part II limit ourselves to outlining the principal arguments rather than giving detailed proofs.

By definition we have $A(p) = e_0/e_i$, or writing $e_i = e_s$,

$$A(p) = (e_0/e_1)(e_1/e_2)\cdots(e_{s-2}/e_{s-1})(e_{s-1}/e_s), \tag{4.2}$$

where e_j represents the voltage across the j-th pair of terminals in Figure 3. Referring to the same figure, we see that e_{j-1}/e_j, $(j = 1, 2, \cdots, s)$ may be interpreted as the transfer function of an L-network whose series arm is Z_{ja} and whose shunt arm consists of Z_{jb} in parallel with the $(j-1)$-stage ladder network having the terminal pairs $0, 1, \cdots, j-1$. Let us denote this L-network by L_j, its shunt arm by Z'_{jb}, and its transfer function by $A_j(p)$, whence $A_j(p) = e_{j-1}/e_j$. It is also convenient to denote the driving point impedance seen at the j-th terminal pair by Z'_{ja} so that

$$Z'_{ja} = Z_{ja} + Z'_{jb}, \qquad (j = 1, 2, \cdots, s). \tag{4.3}$$

If we replace the e_{j-1}/e_j in (4.2) by their equivalent transfer functions, we obtain

$$A(p) = A_1(p)\cdot A_2(p)\cdots A_s(p). \tag{4.4}$$

We shall refer to the $A_j(p)$, $(j = 1, 2, \cdots, s)$ in (4.4) as the *component transfer functions* of $A(p)$ and of the s-stage ladder network. For induction proofs, it is useful to note that $\Pi_{j=1}^{m} A_j(p)$ where $1 \leq m \leq s$ represents the transfer function of the ladder network consisting of the first m-stages in Figure 3. Since each $A_j(p)$, $(j = 1, 2, \cdots, s)$ is the transfer function of the L-network L_j, it immediately follows from (4.4) and the results of Part I that the poles of $A(p)$ are negative while its zeros are non-positive real numbers. However, as in the case of the L-network, the poles and zeros of $A(p)$ must also obey certain order relations which we now proceed to obtain.

By an easy induction we can show that

$$Z'_{ja} = N_j/D_j, \qquad (j = 1, 2, \cdots, s), \tag{4.5}$$

where N_j and D_j are polynomials in p which are to be calculated recursively by means of

$$\begin{cases} N_j = (\eta_j P_j T_j + \lambda_j Q_j R_j) N_{j-1} + \eta_j \lambda_j P_j R_j D_{j-1} \\ D_j = Q_j T_j N_{j-1} + \lambda_j Q_j R_j D_{j-1} \end{cases} \qquad (j = 1, 2, \cdots, s), \tag{4.6}$$

if we define $N_0 = 1$, $D_0 = 0$. This result is now used to obtain a more explicit expression for $A(p)$.

The L-network L_j has Z_{ja} as its series arm and Z'_{jb} as its shunt arm, whence by (1.4) and (4.3) we have

$$A_j(p) = Z'_{jb}/(Z_{ja} + Z'_{jb}) = Z'_{jb}/Z'_{ja}. \tag{4.7}$$

Now since Z'_{jb} consists of Z_{jb} in parallel with $Z'_{(j-1)a}$ we have

$$Z'_{jb} = Z'_{(j-1)a} Z_{jb}/(Z'_{(j-1)a} + Z_{jb}) = \lambda_j Q_j R_j N_{j-1}/D_j, \tag{4.8}$$

where we have used (4.5) and (4.6) in getting the last fraction. Substitution of (4.8) in (4.7) and use of (4.5) then gives us the following expression for $A_j(p)$ as the quotient of two polynomials:

$$A_j(p) = \lambda_j Q_j R_j N_{j-1}/N_j\,, \qquad (j = 1, 2, \cdots, s). \tag{4.9}$$

Equation (4.4) now becomes

$$A(p) = \lambda_1 \frac{Q_1 R_1}{N_1} \cdot \lambda_2 \frac{Q_2 R_2 N_1}{N_2} \cdot \cdots \cdot \lambda_s \frac{Q_s R_s N_{s-1}}{N_s}, \tag{4.10}$$

or

$$A(p) = (1/N_s)\Pi_{i=1}^{s} \lambda_i Q_i R_i\,. \tag{4.11}$$

For simplicity we will suppose that numerator and denominator in (4.9) have no common factor so that the zeros and poles of $A_j(p)$ are the zeros of $Q_j R_j N_{j-1}$ and N_j respectively. The general case follows from this by a limiting process, as in §1 of Part I, and this does not affect the results. Since the poles of $A(p)$ are the poles of the L-network transfer function $A_s(p)$ they are all distinct. Let us designate them by $-\gamma_1\,, -\gamma_2\,, \cdots, -\gamma_M$ where $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_M$. Also denote the zeros of $A(p)$ by $-\delta_1\,, -\delta_2\,, \cdots, -\delta_N$ where $0 \leq \delta_1 \leq \delta_2 \leq \cdots \leq \delta_N$. We require the following further notation:

(i) The sequence of δ's and γ's arranged in order of ascending order of magnitude will be denoted by S_s^*;

(ii) The sequence of zeros and poles of $A_j(p)$ with their signs changed and arranged in ascending order of magnitude will be denoted by $S_j (j = 1, 2, \cdots, s)$;[22]

(iii) Let ξ denote any term in the sequence S_s^*. Then the *excess* $E(\xi)$ is defined as the integer which equals the number of poles up to and including ξ minus the number of zeros up to and including ξ in S_s^*. Similarly, $E(S_s^*)$ is defined as the algebraic difference between the number of poles in S_s^* and the number of zeros in S_s^*.

With regard to the $S_j\,, (j = 1, 2, \cdots, s)$, we note the following two properties: (a) Each S_j has the structure indicated in Theorem 1 for S. We shall refer to an S_j having this property as an L-sequence. (b) By (4.9) the poles in $S_j, (1 \leq j < s)$ coincide with a subset of the zeros in S_{j+1}. Whenever we shall refer in the sequel to sequences $S_j (j = 1, 2, \cdots, s)$, it is to be understood that the S_j have properties (a) and (b).

Equation (4.10) exhibits the process by which one arrives at the sequence S_s^* when the sequences $S_j\,, (j = 1, 2, \cdots, s)$ are given. Write down the sequence S_1. Insert the terms of S_2 into S_1 in their correct position according to numerical magnitude. If a number of zeros of S_2 coincide with a number of zeros of S_1 we make the convention of arranging these in any order which maintains the original orders in S_1 and S_2. Whenever a zero of S_2 coincides with a pole of S_1 delete both terms. In this way all the poles of S_1 will disappear. Now

[22] For convenience, we shall refer to the terms of S_s^* and S_j as the zeros and poles of these sequences although the negatives of these terms are the actual zeros and poles of the corresponding transfer functions $A(p)$ and $A_j(p)$.

insert the terms of S_2 into the sequence resulting from the preceding step and repeat the procedure of deleting equal zeros and poles, etc. Finally we get S_s^*. A numerical example of this process will be found in §6.

We shall speak of S_s^* as the *product* of the sequences S_j, $(j = 1, 2, \cdots, s)$ meaning thereby that the above algorithm has been performed with the S_j to obtain S_s^*. We write

$$S_s^* = S_1 \times S_2 \times \cdots \times S_s . \tag{4.12}$$

The S_j in (4.12) will be referred to as the *components* of S_s^*.

By means of the algorithm just described and using an induction on the number of stages s, we can prove the following theorem which describes the order relations of the poles and zeros of an s-stage ladder.

Theorem 3. *The sequence S_s^* has the following properties:*

$$0 \leq E(S_s^*) \leq s ,$$

$$-s \leq E(\delta_i) \leq s - 1 , \qquad (i = 1, 2, \cdots, N),$$

$$-(s - 1) \leq E(\gamma_j) \leq s, \qquad (j = 1, 2, \cdots, M).$$

We omit the details of the proof which is straightforward.

Let a sequence S_s^* consist of positive, distinct "poles" γ_j, $(j = 1, 2, \cdots, M)$ and non-negative "zeros" δ_i, $(i = 1, 2, \cdots, N)$ arranged in ascending order of magnitude with no $\delta_i = \gamma_j$. Let Theorem 3 hold for S_s^*. For simplicity we shall refer to a sequence S_s^* having these properties as an *s-stage ladder sequence*. It follows that a 1-stage ladder sequence is an L-sequence. This nomenclature is justified by the results of §§4, 5.

5. The Realizability Theorem. We consider now the sufficiency of the conditions in Theorem 3; namely the question of the existence of an s-stage ladder network, the zeros and poles of whose transfer function correspond to the zeros and poles of a given s-stage ladder sequence S_s^*. The answer is affirmative; in fact, there are many such networks.

To show this we first prove that any s-stage ladder sequence S_s^* may be decomposed into a product (4.12) of s components S_j, $(j = 1, 2, \cdots, s)$. We omit the rather lengthy proof which again depends on an induction. In practice one can easily determine many such decompositions by trial. Now consider the function

$$A(p) = K\Pi_{i=1}^{N}(p + \delta_i)/\Pi_{j=1}^{M}(p + \gamma_j) , \tag{5.1}$$

where $K > 0$ and the δ_i and γ_j form an s-stage ladder sequence S_s^*. Let $S_s^* = S_1 \times S_2 \times \cdots \times S_s$ be one decomposition of S_s^* into L-network sequences S_j. Corresponding to each sequence S_j form the rational function $A_j(p)$ whose zeros and poles correspond to the zeros and poles of S_j and whose constant multiplicative factor will be denoted by K_j. Thus $A(p) = \Pi_{j=1}^{s} A_j(p)$ if $K = \Pi_{j=1}^{s} K_j$.

Then we can show that for a suitable choice of the K_j there exists an s-stage

ladder network having $A(p)$ as its transfer function and the $A_j(p)$ as its component transfer functions. In fact we can completely determine the range of values of the K_j (and consequently of K) for which such a network is possible. Evidently, since the $A_j(p)$ are to represent transfer functions of L-networks, we must have $0 < K_j \leq K_{0j}$, where K_{0j} is a number determined from the sequence S_j in the same manner that K_0 is determined from S in Theorem 2. But this necessary condition turns out to be sufficient as well, for with what precedes to be understood, we may state

Theorem 4. *The necessary and sufficient condition that a function* $A(p)$ *of the form* (5.1) *be the transfer function of an* s*-stage ladder network having the* $A_j(p)$, $(j = 1, 2, \cdots, s)$ *as component transfer functions for some choice of the* K_j, $(j = 1, 2, \cdots, s)$ *is that* $0 < K \leq \Pi_{i=1}^{s} K_{0j}$.

In the next section we will, by means of an example, indicate the lines along which a proof of this theorem may be made.

In Theorem 4 we have determined the maximum realizable K for a preassigned, fixed decomposition of S_s^* into s components S_j. One might next ask for the maximum realizable K if we allow all possible decompositions of S_s^* into exactly s components S_j. Except for the case of the L-network, $(s = 1)$, which was treated in Part I, this problem is unsolved as far as we know. The next step would be to remove the restriction on the number of components S_j corresponding to the fact that if s has the smallest value possible for the given sequence of δ_i and γ_j then S_s^* is also an $S_{s+n}^*(n = 1, 2, \cdots)$. This is equivalent to asking for the conditions under which (5.1) is realizable as the transfer function of a ladder network of any number of stages. There is now no restriction on the δ_i and γ_j other than that the γ_j be distinct and positive and that the δ_i be non-negative. As for the maximum realizable K, we have obtained a number of results concerning this question of which one is stated below. The proof as well as a more complete treatment of the whole problem must be deferred.

Consider the transfer function $A(p)$ given by (5.1) and let S^* denote the sequence of δ_i and γ_j. By selecting zeros and poles from S^* form a set of L-sequences $\bar{S}_j$ such that every zero and pole of S^* is in one of the $\bar{S}_j$. Let $\bar{K}_{0j}$ be the maximum K associated with the L-sequence $\bar{S}_j$ and write $l_1 = \Pi \bar{K}_{0j}$. In the same way for each one of the finite number of decompositions of S^* into $\bar{S}_j$ we have an l_i. Now denote by $\bar{K}_0$ the least upper bound of the values of K in $A(p)$ corresponding to the possible decompositions of S^*. Then we have

Theorem: $\bar{K}_0$ *is one of the three quantities*

$$\text{Max } l_i, \ \Pi_{j=1}^{M} \gamma_j / \Pi_{i=1}^{N} \delta_i, \ 1$$

if $N = M$ *and is one of the first two quantities if* $N < M$.

This theorem together with some elementary considerations serves to determine $\bar{K}_0$ in a wide class of cases.

6. An illustrative example. The following example will illustrate the method of synthesizing a ladder network from a given transfer function $A(p)$ of the form (5.1). By generalizing the argument and by giving a detailed treatment

of the various cases which may arise, this procedure may be converted into a proof of Theorem 4.

Let

$$A(p) = K\frac{(p+24)(p+30)(p+40)}{(p+2)(p+9)(p+22)}.$$

Then for the sequence S_s^* we have

$$S_s^*: \begin{matrix}\gamma_1 & \gamma_2 & \gamma_3 & \delta_1 & \delta_2 & \delta_3\\ 2 & 9 & 22 & 24 & 30 & 40\end{matrix}.$$

Since $|E(\xi)| \leq 3$ for ξ any δ or γ of S_s^*, S_s^* is at least a 3-stage ladder network sequence. Let us choose $s = 3$ and consider the following decomposition of $S_s^* = S_1 \times S_2 \times S_3$ where[23]

$$S_3: \begin{matrix}\gamma_{31} & \delta_{31} & & \delta_{32} & \gamma_{32} & \gamma_{33} & & \delta_{33}\\ 2 & 4 & & 8 & 9 & 22 & & 30\end{matrix}$$

$$S_2: \begin{matrix}\gamma_{21} & \delta_{21} & \gamma_{22} & & \delta_{22}\\ 4 & 6 & 8 & & 24\end{matrix}$$

$$S_1: \begin{matrix}\gamma_{11} & & \delta_{11}\\ 6 & & 40\end{matrix}.$$

The corresponding component transfer functions are

$$A_3(p) = K_3\frac{(p+4)(p+8)(p+30)}{(p+2)(p+9)(p+22)},$$

$$A_2(p) = K_2\frac{(p+6)(p+24)}{(p+4)(p+8)}, \qquad A_1(p) = K_1\frac{(p+40)}{(p+6)}.$$

For each of these a K_{0j} is determined following the procedure given in section 3. Since S_1 and S_2 do not contain consecutive γ's we have

$$K_{01} = \text{Min}\left[1, \frac{6}{40}\right] = \frac{3}{20}, \qquad K_{02} = \text{Min}\left[1, \frac{4\times 8}{6\times 24}\right] = \frac{2}{9}.$$

As for S_3 it terminates with a δ so that

$$K_{03} = \text{Min}\left[K_d, \frac{2\times 9\times 22}{4\times 8\times 30} = \frac{33}{80}\right]$$

The value of K_d need not be calculated in this case. For with the value $K_3 = 33/80$ in equation (3.2), the roots of the latter are readily located and found to have the correct order relations with respect to the δ_{3j}. Hence $K_{03} = 33/80$.

For simplicity, let $K_j = K_{0j}$ in each case so that $K = \Pi K_j = 11/800$. Then solving (3.2) with $K = K_{0j}$, $(j = 1, 2, 3)$ respectively we obtain

$$\zeta_{31} = 0, \qquad \zeta_{32} = 10.08, \qquad \zeta_{33} = 16.60;$$

$$\zeta_{21} = 0, \qquad \zeta_{22} = 6.857; \qquad \zeta_{11} = 0.$$

[23] The double subscript notation here and elsewhere in this section is self-explanatory.

These combined with the δ_{jk} give the following set of sequences Σ_j' :

$$\Sigma_3^1 : (0, \bar{4}), (\bar{8}, 10.08), (16.60, 30);$$
$$\Sigma_2^1 : (0, \bar{6}), (6.857, 24); \quad \Sigma_1' : (0, 40).$$

In view of (4.7), we could now proceed to obtain the zeros and poles of Z_{ja} and Z_{jb}' from the Σ_j' by the identification procedure of §3, but we would then find that it is not possible to determine impedances Z_{jk} such that $Z'_{(j+1)b}$ is a parallel combination of Z_{jk} and $Z'_{ja} = Z_{ja} + Z'_{jb}$. However, it can be shown that by modifying the sequences Σ_j' and by properly identifying their terms, we can always obtain Z_{ja} and Z_{jb}' which lead to impedances Z_{jk}. The principal modification is indicated in our example.[24]

In the sequences Σ_j' above we have indicated ζ, δ pairs by parentheses, and we have marked with bars those δ's which coincide with poles of the preceding S sequence. The presence of successive barred zeros in a Σ_j' requires the insertion of a pair (δ, ζ), with $\delta = \zeta$, between these zeros. We have such a pair in Σ_3' where $\delta_{31} = \bar{4}$, $\delta_{32} = \bar{8}$. Let us choose the new pair as (6, 6). Then the modified Σ_j' which we denote by Σ_j are

$$\Sigma_3 : (0, 4), (6, 6), (8, 10.08), (16.60, 30);$$
$$\Sigma_2 : (0, 6), (6.857, 24); \quad \Sigma_1 : (0, 40).$$

Using the method of §3 and the notation $Z_{jb}' = \lambda_j'(p + \beta_{j2}') \cdots /(p + \beta_{j1}') \cdots$ we get as one identification of the Σ_j the following:

$\beta_{31}' = 0, \beta_{32}' = 4, \alpha_{31} = 6, \beta_{33}' = 6, \beta_{34}' = 8, \alpha_{32} = 10.08, \beta_{35}' = 16.60, \beta_{36}' = 30;$
$\beta_{21}' = 0, \beta_{22}' = 6, \beta_{23}' = 6.857, \beta_{24}' = 24; \beta_{11}' = 0, \beta_{12}' = 40.$

The choice of this particular identification enables us to go through with the rest of the syntheses procedure. Rules can be given, which in each case permit us to make an acceptable identification.

From equation (2.3) we find

$$\eta_3/\lambda_3' = (1/K_3) - 1 = \tfrac{40}{33} - 1 = \tfrac{47}{33};$$
$$\eta_2/\lambda_2' = (1/K_2) - 1 = \tfrac{8}{7} - 1 = \tfrac{7}{7};$$
$$\eta_1/\lambda_1' = (1/K_1) - 1 = \tfrac{20}{3} - 1 = \tfrac{17}{3}.$$

We can now write down Z_{ja}, Z_{jb}' and also $Z_{ja}' = Z_{jb}'/A_j(p)$. The latter is required only for $j = 1, 2$. Expressing the multiplicative constants in terms of the λ_j', we have

$$Z_{1a} = \tfrac{17}{3}\lambda_1' \qquad Z_{2a} = \tfrac{7}{7}\lambda_2'$$

$$Z_{1b}' = \lambda_1' \frac{(p + 40)}{p} \qquad Z_{2b}' = \lambda_2' \frac{(p + 6)(p + 24)}{p(p + 6.857)}$$

$$Z_{1a}' = \frac{20\lambda_1'(p + 6)}{3p} \qquad Z_{2a}' = \frac{9\lambda_2'(p + 4)(p + 8)}{2p(p + 6.857)}$$

$$Z_{3a} = \frac{47\lambda_3'(p + 10.08)}{33(p + 6)}, \qquad Z_{3b}' = \frac{\lambda_3'(p + 4)(p + 8)(p + 30)}{p(p + 6)(p + 16.60)}.$$

[24] The only other modification required, concerns the insertion of the pair (0, 0) into Σ_j' under certain conditions.

Take $Z_{1b} = Z'_{1b}$ and determine Y_{jb}, $(j = 2, 3)$ from the equation

$$Y_{jb} = Y'_{jb} - Y'_{(j-1)a}$$

where Y'_{jb} and $Y'_{(j-1)a}$ are the reciprocals of Z'_{jb} and $Z'_{(j-1)a}$ respectively. After expansion of the admittances on the right into their canonical forms and consolidation of terms, there results

$$Y_{2b} = \left[\frac{.0476}{\lambda'_2} - \frac{3}{20\lambda'_1}\right]\frac{p}{(p+6)} + \frac{.9524\,p}{\lambda'_2(p+24)};$$

$$Y_{3b} = \left[\frac{.2423}{\lambda'_3} - \frac{1.4286}{9\lambda'_2}\right]\frac{p}{(p+4)} + \left[\frac{.1955}{\lambda'_3} - \frac{.5716}{9\lambda'_2}\right]\frac{p}{(p+8)} + \frac{.5622p}{\lambda'_3(p+30)}.$$

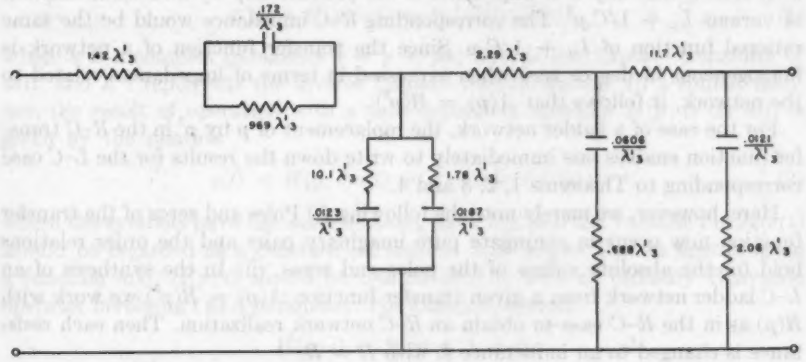

FIGURE 4

Each of the brackets on the right in these equations must be non-negative if Y_{jb}, $(j = 2, 3)$ is to be an R–C admittance function. Hence

$$\lambda'_1 \geq 3.15\lambda'_2$$

and

$$\lambda'_2 \geq \text{Max}\,[.3249, .6551] \times \lambda'_3 = .6551\lambda'_3.$$

Let us choose the equality sign in each of these, since this will eliminate a term in each of Y_{jb}, $(j = 2, 3)$ and hence yield the simplest networks. Then we get

$$Y_{2b} = \frac{1}{\lambda'_2}\left[\frac{.0986p}{p+8} + \frac{.5622p}{p+30}\right], \qquad Y_{3b} = \frac{1.454p}{\lambda'_3(p+24)}.$$

Finally getting Foster realizations of Z_{1a}, Z_{1b}, Z_{2a}, Y_{2b}, Z_{3a}, Y_{3b} and expressing all element values in terms of λ'_3 we arrive at the following network as a realization of $A(p)$:

7. The R–L and L–C cases.[25] Let $A(p)$ denote the transfer function of a four terminal R–L network. If each inductance L of the network is replaced by a condenser C with $C = 1/L$ while each resistor remains unchanged, we obtain

[25] In this section we allow the use of self-inductance only.

a corresponding R–C network. Let $B(p)$ denote the transfer function of this network. Since the impedance Lp of an inductance has been changed to the impedance $1/Cp = L/p$ of a condenser, it is evident that $B(p) = A(1/p)$ or $A(p) = B(1/p)$.

For the case of an R–C ladder network, the substitution of $1/p$ for p in the transfer function leads to a function of the same type. Hence Theorems 1 and 2 hold for an R–L ladder network. It is also readily seen by the use of this same transformation that Theorems 3 and 4 are valid for the R–L case.

The L–C case is treated similarly. To each four terminal L–C network having transfer function $A(p)$ there corresponds an R–C network with transfer function $B(p)$ obtained by replacing each inductance L by a resistance R with $R = L$. It is known that any L–C impedance is the product of p by a rational function of various $L_i + 1/C_i p^2$. The corresponding R–C impedance would be the same rational function of $L_i + 1/C_i p$. Since the transfer function of a network is homogeneous of degree zero when expressed in terms of impedances related to the network, it follows that $A(p) = B(p^2)$.

For the case of a ladder network, the replacement of p by p^2 in the R–C transfer function enables one immediately to write down the results for the L–C case corresponding to Theorems 1, 2, 3 and 4.

Here, however, we merely note the following (i) Poles and zeros of the transfer function now occur in conjugate pure imaginary pairs and the order relations hold for the absolute values of the poles and zeros. (ii) In the synthesis of an L–C ladder network from a given transfer function $A(p) = B(p^2)$ we work with $B(p)$ as in the R–C case to obtain an R–C network realization. Then each resistance is changed to an inductance L with $L = R$.

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TIME-DEPENDENT HEAVISIDE OPERATORS

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1. Introduction. The present paper is devoted to a brief and formal discussion of some of the basic properties of so-called time-dependent Heaviside operators. The results presented here stem largely from our earlier work on linear varying-parameter systems.[1]

The concept of a time-dependent Heaviside operator $H(p;t)$ is essentially a generalization of that of a conventional Heaviside operator $H(p)$. Thus, $H(p)$ operates on a function $u(t)$ ($u(t) = 0$ for $t < 0$) in accordance with the relation

$$v(t) = H(p)u(t) = \mathfrak{L}^{-1}\{H(s)U(s)\} \tag{1}$$

where s is a complex parameter $s = \sigma + jw$, $U(s)$ is the Laplace transform of $u(t)$, and $\mathfrak{L}^{-1}$ represents the inverse Laplace transformation. In a similar manner, the result of operation with a time-dependent operator $H(p;t)$ on $u(t)$ is given by the relation

$$v(t) = H(p;t)u(t) = \mathfrak{L}^{-1}\{H(s;t)U(s)\} \tag{2}$$

where the symbols have the same meaning as in (1), and the variable t in $H(s;t)$ should be regarded as a constant parameter. This means that in operating on a function $u(t)$, $H(p;t)$ should be treated as if it were an ordinary Heaviside operator involving t as a parameter. For example, assume

$$H(p;t) = 1/(p + 1 + t)$$

and $u(t) = 1(t)$ (unit-step function). For this case $U(s) = 1/s$, and hence

$$v(t) = \mathfrak{L}^{-1}\{1/s(s + 1 + t)\}. \tag{3}$$

Treating t in this expression as a constant and using a table of Laplace transforms, we find

$$v(t) = [1/(1 + t)](1 - e^{-(1+t)t}), \qquad t \geqq 0. \tag{4}$$

It will be noted that when $H(p;t)$ is a polynomial in p, p plays the role of d/dt, that is, $H(p;t)$ reduces to a differential operator. When $H(p;t)$ is analytic in the neighborhood of $p = 0$, it may be expressed as a power series in p and hence is equivalent to a differential operator of infinite order. Operators of this type have been extensively investigated by Davis.[2]

In the sequel there will be occasions when $H(p;t)$ will be referred to as a *bilateral* time-dependent Heaviside operator, or simply a bilateral operator. In such cases $u(t) \neq 0$ for $t < 0$ and the quantity $U(s)$ in (2) should be interpreted as the bilateral Laplace transform of $u(t)$.

[1] In particular, Proc. I. R. E., **38**, pp. 291–299 (1950); and Jour. App. Phys., **21**, 1171–1177 (1950).

[2] H. T. Davis, *The Theory of Linear Operators (from the Standpoint of Differential Equations of Infinite Order)*, Principia Press, 1936.

2. System Function. In this section we shall establish a basic connection between the so-called system function[1] of a linear varying-parameter system and the concept of a time-dependent Heaviside operator.

Let N be a linear varying-parameter system, that is, a linear system in which one or more parameter-values are specified functions of time. Let $u(t)$ be the input to N and let $v(t)$ be the corresponding output, i.e., the response of N to $u(t)$. The system is assumed to be at rest (but not necessarily fixed) prior to the application of the input.

The system function of N is defined by the relation

$$H(jw; t) = v(t)/u(t) \mid u(t) = e^{jwt}, \tag{5}$$

which means that $H(jw; t)e^{jwt}$ represents the response of N to e^{jwt}.

The fundamental property of the system function may be stated as follows.

THEOREM 1: *If $H(jw; t)$ is the system function of a linear system N, then the response of N to an input $u(t)$ is*

$$v(t) = H(p; t)u(t) \tag{6}$$

where $v(t)$ is the response to $u(t)$, and $H(p; t)$ should be treated as a time-dependent Heaviside operator.

This property of the system function is a natural consequence of: (a) the additive property of linear systems; (b) the resolution of $u(t)$ in terms of components of the form $e^{st}U(s)ds$ through the use of Laplace transformation; and (c) the definition of the system function to the effect that $H(jw; t)e^{jwt}$ is the response to e^{jwt} or, equivalently, $H(s; t)e^{st}$ is the response to e^{st}.

The stated theorem can readily be proved by starting with a specific description of N and then making use of the definition of the system function. A convenient starting point is provided by the basic relation

$$v(t) = \int_0^\infty W(t, \tau)u(t - \tau)\, d\tau \tag{7}$$

where $W(t, \tau)$ represents the so-called *impulsive response* of N, i.e., the response of N at the time t to a unit impulse applied at the time $t - \tau$. $W(t, \tau)$ is essentially an integral operator relating $v(t)$ to $u(t)$.

Setting $u(t) = e^{st}$ and $v(t) = H(s; t)e^{st}$ in equation (7), we obtain

$$H(s; t)e^{st} = \int_0^\infty W(t, \tau)e^{s(t-\tau)}\, d\tau \tag{8}$$

$$H(s; t) = \int_0^\infty W(t, \tau)e^{-s\tau}\, d\tau. \tag{9}$$

From this it follows that $H(s; t)$ and $W(t, \tau)$ constitute a Laplace transform pair involving t as a parameter, i.e.,

$$H(s; t) = \mathfrak{L}\{W(t, \tau)\} \tag{10}$$

and

$$W(t, \tau) = \mathfrak{L}^{-1}\{H(s; t)\}. \tag{11}$$

Next, expressing $u(t)$ in terms of its Laplace transform $U(s)$,

$$u(t) = 1/2\pi j \int_{c-j\infty}^{c+j\infty} U(s)e^{st}\,ds, \tag{12}$$

and substituting this in (7), we get

$$v(t) = 1/2\pi j \int_0^\infty d\tau W(t,\tau) \int_{c-j\infty}^{c+j\infty} U(s)\, e^{s(t-\tau)}\,ds \tag{13}$$

which upon inversion of the order of integration becomes

$$v(t) = 1/2\pi j \int_{c-j\infty}^{c+j\infty} ds\, e^{st}\, U(s) \int_0^\infty W(t,\tau)\, e^{-s\tau}\,d\tau. \tag{14}$$

Making use of (9), (14) reduces to

$$v(t) = 1/2\pi j \int_{c-j\infty}^{c+j\infty} H(s;t)\, U(s)\, e^{st}\,ds \tag{15}$$

or

$$v(t) = \mathfrak{L}^{-1}\{H(s;t)U(s)\}. \tag{16}$$

In view of the definition of $H(p;t)$ (cf. (2)), (16) may equivalently be written

$$v(t) = H(p;t)u(t) \tag{17}$$

which thus formally establishes the theorem.

3. Product Relation. In the case of conventional Heaviside operators the product relation may be written as

$$H_2(p)\{H_1(p)u(t)\} \equiv H_3(p)u(t) \tag{18}$$

where the operator $H_3(p)$ is equal to the algebraic product of $H_2(p)$ and $H_1(p)$, i.e.

$$H_3(p) = H_2(p)H_1(p). \tag{19}$$

In the case of time-dependent operators, the product relation assumes the following form

$$H_2(p;t)\{H_1(p;t)u(t)\} \equiv H_3(p;t)u(t) \tag{20}$$

where $H_3(p;t)$—the product of $H_2(p;t)$ and $H_1(p;t)$—is given by the operational relation

$$H_3(s;t) = H_2(p+s;t)H_1(s;t) \tag{21}$$

in which s plays the role of a parameter, $H_1(s;t)$ is the operand, and $H_2(p+s;t)$ should be treated as a bilateral time-dependent operator. This relation may readily be deduced from (20) by substituting e^{st} for $u(t)$. Equation (21) reduces to the algebraic relation (19) when $H_2(p;t)$ and $H_1(p;t)$ are independent of t.

It will be convenient to express (21) in a symbolic form

$$H_3 = H_2 * H_1. \tag{22}$$

The operation $*$ is distributive, associative and, in general, non-commutative. Thus,

$$H_a * (H_b + H_c) \equiv H_a * H_b + H_a * H_c \tag{23}$$

$$H_a * (H_b * H_c) \equiv (H_a * H_b) * H_c, \tag{24}$$

and generally $H_a * H_b \neq H_b * H_a$.

The inverse of an operator $H(p; t)$, or simply H, is denoted by $H^{-1}(p; t)$, or simply H^{-1}, and is defined by the relations

$$H * H^{-1} = H^{-1} * H = 1 \tag{25}$$

or more explicitly

$$H(p + s; t)H^{-1}(s; t) = H^{-1}(p + s; t)H(s; t) = 1. \tag{26}$$

In the preceding section it was shown that the relation

$$v(t) = \int_0^\infty W(t, \tau)u(t - \tau)\, d\tau \tag{27}$$

may equivalently be written in the form

$$v(t) = H(p; t)u(t), \tag{28}$$

and that $W(t, \tau)$ and $H(s; t)$ constitute a Laplace transform pair involving t as a parameter. It can readily be verified that if $H_3 = H_2 * H_1$, then the corresponding integral operators W_3, W_2, and W_1 are related to each other by

$$W_3(t, \tau) = \int_0^\infty W_2(t, \tau')W_1(t - \tau', \tau - \tau')\, d\tau' \tag{29}$$

which is equivalent to Volterra's rule of composition.[3]

4. Differential Equations. A particular and especially important case is one where $v(t)$ and $u(t)$ are related to each other by a differential equation of the form

$$L(p; t)v(t) = K(p; t)u(t) \tag{30}$$

where $L(p; t)$ and $K(p; t)$ are polynomials in p, i.e. differential operators. Physically, this case arises whenever the linear system of which $u(t)$ and $v(t)$ are respectively the input and the output, is of the "lumped-constant" type.

Equation (30) may equivalently be written

$$v(t) = H(p; t)u(t) \tag{31}$$

where $H = L^{-1} * K$. From this it follows that H satisfies the differential equation

$$L * H = K \tag{32}$$

or more explicitly

$$L(p + s; t)H(s; t) = K(s; t). \tag{33}$$

[3] V. Volterra, *Theory of Functionals*, Blackie & Son, Ltd., London, 1930.

As was mentioned above, equation (30) may be regarded as a relation between the input and output of a linear "lumped-constant" system. The system function $H(s; t)$ of this system is the steady-state solution of (33). In many practical cases, and particularly in the case of slowly-varying systems, a convenient approximate method for determining the response[4] to a given input (i.e. given $u(t)$, solving (30) for $v(t)$) would consist in deriving an approximate expression for $H(s; t)$ from (33), and then using (2) in conjunction with a table of Laplace transforms to obtain $v(t)$.

5. Stochastic Heaviside Operators. In the foregoing discussion we were concerned with operators which are non-random functions of time. More generally, however, $H(p; t)$ may be a random function of time involving p as a parameter. For convenience in terminology, operators of this type will be referred to as *stochastic Heaviside operators*, or simply, stochastic operators. A special class of stochastic Heaviside operators is one in which $H(p; t)$ is a polynomial in p. This class of operators (stochastic *differential* operators) has been extensively investigated[5] in connection with stochastic differential equations.

When the operator $H(p; t)$ or the operand $u(t)$, or both, are random functions, it is expedient to express the rule of operation with $H(p; t)$ on $u(t)$ in the following form

$$v(t) = H(p; t)u(t) = \int_{-\infty}^{\infty} H(jw; t)e^{jwt}\, dU(jw) \tag{34}$$

where $U(jw)$ is the Fourier-Stieltjes transform of $u(t)$. The reason for using the Stieltjes integral formulation is that a much wider class of stochastic processes can be represented in the Fourier-Stieltjes form[6]

$$u(t) = \int_{-\infty}^{\infty} e^{jwt}\, dU(jw) \tag{35}$$

than in the usual Fourier integral form.

Consider a stochastic Heaviside operator $H(p; t)$ which is stationary in the wide sense[6]. The *correlation function* of $H(p; t)$ is defined as the time-average of the algebraic product $H(p; t)H(-p; t + \tau)$, i.e.

$$\psi_H(p; \tau) = \overline{H(p; t)H(-p; t + \tau)}. \tag{36}$$

Let $u(t)$ be stationary (wide sense) and independent of $H(p; t)$. Denote the correlation functions of $u(t)$ and $v(t)$ by $\psi_u(\tau)$ and $\psi_v(\tau)$, respectively. The following relation bewteen $\psi_v(\tau)$, $\psi_u(\tau)$, and $\psi_H(p; \tau)$ can readily be established.[7]

[4] The case where the system is not initially at rest is treated in *Initial Conditions in Linear Varying-Parameter Systems*, Jour. App. Phys., **22**, 782–786 (1951).

[5] P. Levy, *Processus Stochastiques et Mouvement Brownien*, Gauthier-Villars, Paris, 1948.

[6] J. L. Doob, *Time Series and Harmonic Analysis*, Proc. Berkeley Symposium on Math. Stat. and Probability, 303–343 (1949).

[7] A proof of this theorem is given in *Correlation Functions and Power Spectra in Variable Networks*, Proc. I. R. E., **38**, 1342–1345 (1949). An application of this theorem is discussed in *Correlation Functions and Spectra of Phase- and Delay-Modulated Signals*, Proc. I. R. E., **39**, 425–428 (1951).

THEOREM 2: *If*

$$v(t) = H(p; t)u(t) \tag{37}$$

where $H(p; t)$ and $u(t)$ are stationary (wide sense) and statistically independent of each other, then the correlation function of $v(t)$ is related to that of $u(t)$ by the operational relation

$$\psi_v(\tau) = \psi_H(p; \tau)\psi_u(\tau) \tag{38}$$

where the operator $\psi_H(p; \tau)$ is the correlation function of $H(p; t)$.

This theorem has many applications, particularly in connection with the harmonic analysis of stationary stochastic processes.

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DRAG OF A FINITE WEDGE AT HIGH SUBSONIC SPEEDS*

By Julian D. Cole

1. Introduction. In this paper a method of approximately finding the pressure drag of a finite wedge at subsonic free stream Mach numbers close to one is discussed. The flow about the wedge profile (Fig. 1) is investigated by means of the classical methods of gas dynamics and the transonic approximation is used. The wedge section is chosen for its geometrical simplicity which permits the formulation of a problem with known boundaries in the hodograph plane. In addition important points about the pressure distribution on the wedge are known in advance since the flow field has a stagnation point at the nose and the sonic velocity is reached at the beginning of the corner. In the following work the attention will be mainly on the flow between the nose and the corner. The author wishes to thank H. W. Liepmann and the Transonic Research Group for helpful discussions and experimental data and P. A. Lagerstrom and Th. Wu for useful criticism.

2. Equations of Motion in Physical Plane and Hodograph. The equations of motion apply to steady isentropic motion in a perfect, non-viscous gas. The main additional assumption is that of transonic flow; it is assumed that the flow field is perturbed about uniform flow at the sonic velocity:

$$q_1 = a^* + u_1, \qquad q_2 = u_2 \tag{1}$$

where:

q_1 = velocity in x-direction

q_2 = velocity in y-direction

a^* = critical velocity at which sonic speed equals flow speed

and $u_1, u_2 \ll a^*$. In this case the equation of continuity can be approximated by

$$[(\gamma + 1)/a^*]u_1 \, \partial u_1/\partial x - \partial u_2/\partial y = 0 \tag{2}$$

and the condition of irrotationality is

$$\partial u_1/\partial y - \partial u_2/\partial x = 0 \tag{3}$$

The derivation of these equations and similarity laws derivable from them for special problems are discussed in Refs. 1, 2, 3, and 4. These equations are discussed in Refs. 3 and 4 and it is shown that they are qualitatively correct and quantitatively accurate for thin bodies in an appreciable range of Mach numbers close to one. It can also be shown that these equations are approximately valid even if shock waves are present. The rotation introduced is of higher order.

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Within the approximation the Mach number M is given by

$$M^2 - 1 = [(\gamma + 1)/a^*]u_1 \tag{4}$$

It is convenient to introduce the dimensionless variables

$$u = [(\gamma + 1)/a^*]u_1 = -w, \qquad v = [(\gamma + 1)/a^*]u_2 \tag{5}$$

Then Eqs. (2) and (3) give the following equations for two functions $u(x, y)$, $v(x, y)$ in some domain of the (x, y) plane.

$$uu_x - v_y = 0 \qquad u_y - v_x = 0 \tag{6), (7}$$

The system {(6), (7)} is non-linear and of changing type; for $u > 0$, supersonic, the system is hyperbolic and for $u < 0$, subsonic, the system is elliptic. It is the

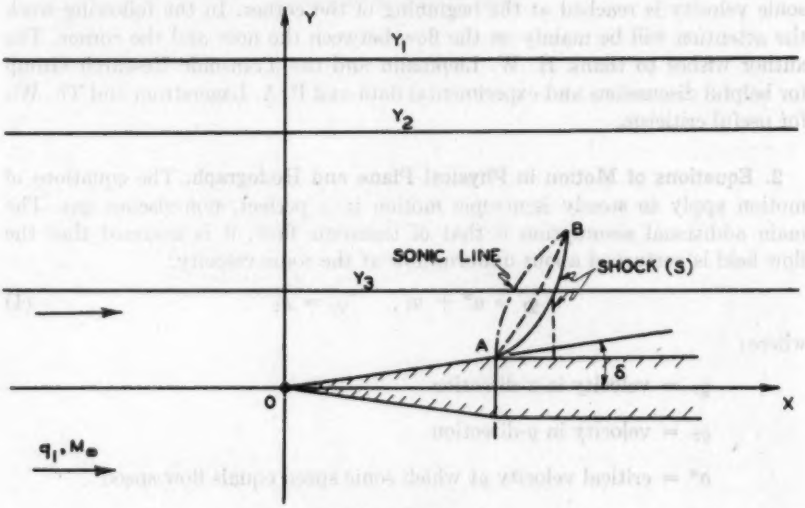

FIG. 1. Physical Plane

simplest system retaining these essential features. The non-linearity prohibits a direct solution of (6) and (7) and it becomes necessary to study the problem in the hodograph.

The system {(6), (7)} becomes linear in the hodograph. A simple way of transforming {(6), (7)} is to consider the spatial coordinates (x, y) as functions of the velocity components as $x(u, v)$, $y(u, v)$. The transformation is given in this case by

$$\partial x/\partial u = (1/J)\partial v/\partial y \qquad \partial y/\partial u = -(1/J)\partial v/\partial x \tag{8}$$

$$\partial x/\partial v = -(1/J)\partial u/\partial y \qquad \partial y/\partial v = (1/J)\partial u/\partial x$$

where

$$J = \text{Jacobian of the transformation} = \partial(u, v)/\partial(x, y) \tag{9}$$

It can be noted that

$$J = u_x v_y - v_x u_y = u u_x^2 - v_x^2 < 0 \quad \text{for} \quad u < 0$$

so that the mapping to the hodograph is locally order reversing in the subsonic region. Under the transformation {(6), (7)} become

$$u y_v - x_u = 0 \qquad x_v - y_u = 0 \qquad (10), (11)$$

Eqs. (10) and (11) form the linear system to be used in the hodograph. Replacement of u by $-w$, and elimination of x, yields

$$w y_{vv} + y_{ww} = 0 \qquad (12)$$

This is the equation studied by Tricomi in Ref. 7. Simple solutions of Eq. (12) obtained by separation of variables are

$$y \sim e^{\pm \lambda v} \sqrt{w} C_{1/3}(\tfrac{2}{3}\lambda w^{3/2}) \qquad (13)$$

where $C_{1/3}$ is any cylindrical or Bessel function of 1/3 order (notation of Ref. 6). It is convenient to use

$$z = \tfrac{2}{3} w^{3/2} \qquad (14)$$

in future work so that

$$y \sim e^{\pm \lambda v} z^{1/3} C_{1/3}(\lambda z) \qquad (15)$$

If the characteristics of system (10), (11) in the hodograph are denoted by

$$X(u, v) = \text{const.} \qquad (16)$$

then the characteristic condition is

$$u X_v^2 - X_u^2 = 0 \qquad (17)$$

On a characteristic

$$X_v / X_u = -du/dv = \pm 1/\sqrt{u}$$

or

$$v = C \pm \tfrac{2}{3} u^{3/2} \qquad (18)$$

where C is a constant.

3. Boundary Conditions. The wedge is assumed to be symmetrical about the x-axis, and to have the semi-angle δ (Fig. 1). The nose is at the origin of coordinates and the wedge has unit length. There is a uniform subsonic flow at upstream infinity directed along the x-axis. This set-up yields the following boundary conditions: Uniform flow at infinity:

$$w = 1 - M_\infty^2 = w_1 = (\tfrac{2}{3} z_1)^{2/3} \qquad \text{at infinity} \qquad (19)$$

Flow tangent to the body: The actual condition is $u_2/(a^* + u_1) = \tan \delta$ on the body surface $y = x \tan \delta$. But this is replaced by the approximate condition

$$v = v_0 = (\gamma + 1)\delta \quad \text{for} \quad y = 0, \qquad 0 < x < 1 \qquad (20)$$

In addition symmetry demands

$$v = 0 \quad \text{at} \quad y = 0 \quad \text{for} \quad x < 0, \qquad x > 1 \tag{21}$$

The same condition (21) applies if the wedge is followed by infinite straight sides parallel to the x-axis. Thus, the problem can be studied in the upper half plane $y \geqq 0$.

It can also be proved (Refs. 3 and 5) that sonic velocity must be reached at the beginning of the corner and that at least one shock wave is necessary in turning the corner. This shock wave (or waves) is demanded by the boundary conditions. Further, in the neighborhood of the nose, the actual stagnation condition is approximated by the stagnation condition of the linearized theory.

$$w \to \infty \quad \text{at the nose} \quad x = 0, \qquad y = 0 \tag{22}$$

This familiar result of thin airfoil theory is a consequence of satisfying the boundary conditions approximately.

The corresponding boundary conditions and the boundary value problem in the hodograph will now be discussed. The hodograph chosen here is (u, v). Consider first the domain into which the flow field is mapped for $y > 0$ if $w > 0$. In that case the equation for v is elliptic, so that

$$0 \leq v \leq v_0, \qquad w > 0 \tag{23}$$

since v attains its maximum and minimum at the boundary. The sonic line is also part of the boundary but it can be shown (Refs. 3 and 5) that in Fig. 1 v decreases traveling along the sonic line away from the front part of the wedge. It also seems plausible that this behavior persists when shock waves are present so that Eq. (23) is taken to hold here. In addition $w \to \infty$ at the stagnation point of the linearized theory. This strip (23) defines the domain for $y > 0$ (Fig. 2). Subsonic points of the physical plane must appear in this region. The boundary conditions (20) and (21) are reproduced in the hodograph, as

$$y = 0 \quad \text{at} \quad v = 0, \qquad w > 0 \tag{24}$$

$$y = 0 \quad \text{at} \quad v = v_0, \qquad w > 0 \tag{25}$$

The linearized stagnation point at the nose means

$$y = 0, \qquad x = 0 \quad \text{as} \quad w \to \infty \tag{26}$$

The fact that sonic velocity is attained at the beginning of the corner in the physical plane fixes the length of the wedge in the hodograph

$$x = 1 \quad \text{at} \quad v = v_0, \qquad w = 0 \tag{27}$$

Further it is assumed that in turning the corner the flow is locally Prandtl-Meyer. Thus the shoulder point maps onto a characteristic in the hodograph. Hence:

$$v = v_0 - \tfrac{2}{3}u^{3/2} \qquad x = 1, \qquad y = 0 \tag{28}$$

It is believed that the supersonic zone is closed with one or more shocks in the physical plane. These shocks appear as gaps in the lines $y(u, v) = \text{constant}$

of the hodograph solution. That is, at the end of a certain jump of (u, v) we must have the same value of (x, y). The jumps of (u, v) are given by the appropriate form of the shock relation in the hodograph variables:

$$v - v_a = (u_a - u)\sqrt{\tfrac{1}{2}(u_a + u)} \tag{29}$$

where u_a, v_a denote the conditions ahead of the shock wave. At present it is believed that these shock waves may be fitted into the main, or subsonic part of the solution, although possibly not uniquely. The exact boundary value problem will be discussed in a later section.

From the homogeneity of the boundary conditions on $y(w, v)$ (24), (25), (26) and (28) it seems plausible that a singularity is necessary to provide a

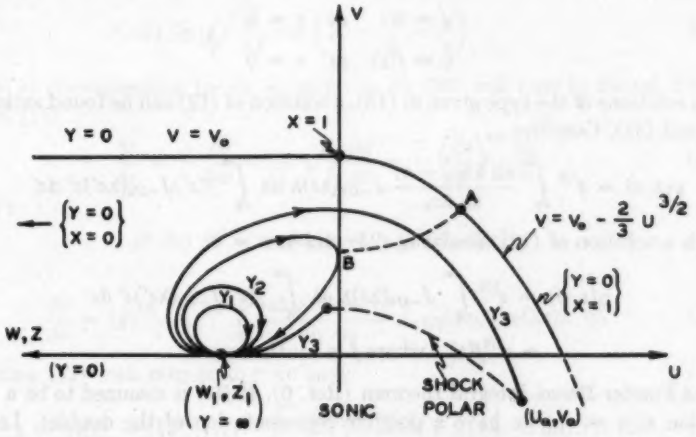

FIG. 2. Hodograph

non-trivial solution of (12) (as in the elliptic case). Such a singularity is the free stream given by (19). The singularity may be characterized by studying the lines y = constant in the physical and hodograph planes. Consider two lines y_1, y_2 = constant $y_1 > y_2$, neither of which passes through the sonic zone. Traveling in the x direction from $(-\infty)$ to $(+\infty)$ on y_1, the flow is first slowed down by the effect of the stagnation point and next speeded up due to the corner, and then returns to the free stream condition. On y_2 the picture is the same except that the effects of the stagnation point and of the corner are each stronger. y_2 thus appears outside y_1 in the hodograph, in agreement with the order reversing property of the mapping described before. These lines and the flow direction on them are shown in Figs. 1 and 2. A line y_3 = constant which passes through the supersonic zone is also sketched. From the behavior of y on the line $v = 0$, namely

$$y = 0 \qquad z \neq z_1, \qquad y = \infty \qquad z = z_1 \tag{30}$$

and from the behavior of the lines y = constant near $z = z_1$, it can be inferred that locally y has a doublet singularity. The representation of the free stream as a doublet singularity can also easily be seen if the corresponding incompressible problem is solved in the hodograph plane. It is clear that the doublet singularity is an important part of the solution to the problem for Eq. (12) defined by boundary conditions (24) through (28). The problem of finding this doublet and the significance of the solution are discussed in the next section.

4. Representation of the Fundamental Doublet. The doublet singularity at $z = z_1$ is considered here in conjunction with boundary condition (25). Both conditions (29) and (25) can be satisfied if first a more general problem is considered, namely

$$y = 0 \quad \text{at} \quad v = v_0 \tag{25}$$

$$y = f(z) \quad \text{at} \quad v = 0 \tag{31}$$

Using solutions of the type given in (15), a solution of (12) can be found satisfying (25) and (31). Consider

$$y(z, v) = z^{1/2} \int_0^\infty \frac{\operatorname{sh} \lambda(v_0 - v)}{\operatorname{sh} \lambda v_0} J_{-1/3}(\lambda z)\lambda \, d\lambda \int_0^\infty f(z')J_{-1/3}(\lambda z')z' \, dz' \tag{32}$$

This is a solution of (12) satisfying (25) and at $v = 0$

$$\begin{aligned} y(z, 0) &= z^{1/2} \int_0^\infty J_{-1/3}(\lambda z)\lambda \, d\lambda \int_0^\infty f(z')J_{-1/3}(\lambda z')z' \, dz' \\ &= z^{1/2}f(z) \quad \text{where } f \text{ is continuous} \end{aligned} \tag{33}$$

by the Fourier-Bessel integral theorem (Ref. 6). If $f(z)$ is assumed to be a delta function at $z = z_1$, we have a possible representation of the doublet. Let the strength of the delta function be A so that

$$f(z) = A\delta(z - z_1) \tag{34}$$

Then (32) yields

$$y(z, v) = Az^{1/2}z_1 \int_0^\infty \frac{\operatorname{sh} \lambda(v_0 - v)}{\operatorname{sh} \lambda v_0} J_{-1/3}(\lambda z)J_{-1/3}(\lambda z_1)\lambda \, d\lambda \tag{35}$$

It can actually be verified, by standard methods, that a fundamental solution of (12) can be represented by

$$\Gamma_1(z, v; z_1) = \tfrac{1}{2}(\tfrac{2}{3}zz_1)^{1/2} \int_0^\infty e^{-\alpha v}J_{-1/3}(\alpha z)J_{-1/3}(\alpha z_1) \, d\alpha \qquad v \geq 0 \tag{36}$$

and the doublet by

$$-\frac{\partial \Gamma_1}{\partial v} = \tfrac{1}{2}(\tfrac{2}{3}zz_1)^{1/2} \int_0^\infty e^{-\alpha v}J_{-1/3}(\alpha z)J_{-1/3}(\alpha z_1)\alpha \, d\alpha \qquad v > 0 \tag{37}$$

Thus (35) can be interpreted as the reflection of infinitely many doublets located so as to satisfy $y = 0$ at $v = v_0$.

It is also possible to form fundamental solutions and doublets from $J_{1/3}$:

$$\Gamma_{II} = -\tfrac{1}{2}(\tfrac{2}{3}zz_1)^{1/3}\int_0^\infty e^{-\alpha v}J_{1/3}(\alpha z)J_{1/3}(\alpha z_1)\,d\alpha \qquad v \geq 0 \tag{38}$$

Solutions like (38) have $(\Gamma_{II}, \partial\Gamma_{II}/\partial v) = 0$ at $z = 0$ and these seem less interesting than solutions like (36), for the approximate method considered here. For more general applications linear combinations of Γ_I and Γ_{II} may be useful. Similar solutions have been given by Carrier and Ehlers, Ref. 11, and Weinstein Ref. 12.*

Now the strength A of the doublet (35) can be adjusted so that the wedge has unit length. It is also possible to satisfy the conditions at the stagnation point. It follows directly from the asymptotic behavior of $J_{-1/3}$ that $y(z, v)$ as given by (35) approaches zero as $z \to \infty$. That is,

$$J_\nu(Z) \cong \sqrt{\frac{2}{\pi Z}}\cos\left(Z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) - \cdots \tag{39}$$

The $x(z, v)$ corresponding to $y(z, v)$ as given by (35) will now be found. From (11) and (14)

$$\frac{\partial x}{\partial v} = -\frac{\partial y}{\partial w} = -\sqrt{w}\,\frac{\partial y}{\partial z} = -\left(\frac{3z}{2}\right)^{1/3}\frac{\partial y}{\partial z} \tag{40}$$

Further:

$$(\partial/\partial z)\,\{z^{1/3}J_{-1/3}(\lambda z)\} = -\lambda z^{1/3}J_{2/3}(\lambda z) \tag{41}$$

Hence

$$\frac{\partial x}{\partial v} = (\tfrac{2}{3})^{1/3}Az_1z^{2/3}\int_0^\infty \frac{\operatorname{sh}\lambda(v_0 - v)}{\operatorname{sh}\lambda v_0}\,J_{-1/3}(\lambda z_1)J_{2/3}(\lambda z)\lambda^2\,d\lambda \tag{42}$$

Integrating (42) with respect to v we have

$$x = x_1(z) - (\tfrac{2}{3})^{1/3}Az_1z^{2/3}\int_0^\infty \frac{\operatorname{ch}\lambda(v_0 - v)}{\operatorname{sh}\lambda v_0}\,J_{-1/3}(\lambda z_1)J_{2/3}(\lambda z)\lambda\,d\lambda \tag{43}$$

It can be shown that x_1 must be constant by considering Eq. (10). The constants x_1, A can now be evaluated. On the surface of the wedge $v = v_0$

$$x(z, v_0; z_1) = x_1 - (\tfrac{2}{3})^{1/3}Az_1z^{2/3}\int_0^\infty \operatorname{csch}\lambda v_0\,J_{-1/3}(\lambda z_1)J_{2/3}(\lambda z)\lambda\,d\lambda \tag{44}$$

As $z \to 0$

$$x(0, v_0; z_1) = x_1 \quad \text{hence} \quad x_1 = 1 \tag{45}$$

It is not clear from (44) that x is finite as $z \to \infty$. Another representation of x is needed in order to study the behavior for large z. A useful series representation of (44) can be obtained by expanding $\operatorname{csch}\lambda v_0$ in partial fractions

$$\operatorname{csch}\lambda v_0 = \frac{1}{\lambda v_0} + 2\sum_{m=1}^{\infty}\frac{(-)^m\lambda v_0}{(\lambda v_0)^2 + (m\pi)^2} \tag{46}$$

* Added in Proof: It has been brought to the author's attention that the fundamental solution (36) of the differential equation (12) has previously been obtained by A. Weinstein as Equation (24) of his paper, *Discontinuous integrals and generalized potential theory*, Trans. Am. Math. Soc., **63**, 342, March 1948.

Substituting (46) in (43) we have

$$x(z, v_0; z_1) = 1 - (\tfrac{3}{2})^{1/3} z_1 A z^{2/3} \int_0^\infty J_{-1/3}(\lambda z_1) J_{2/3}(\lambda z) \cdot \left\{ \frac{1}{v_0} + 2v_0 \sum_{m=1}^{\infty} \frac{(-)^m \lambda^2}{\lambda^2 v_0^2 + m^2\pi^2} \right\} d\lambda \tag{47}$$

Each integral in (47) can be evaluated (Ref. 6, p. 406) and (Ref. 6, p. 430). We have

$$\int_0^\infty J_{-1/3}(\lambda z_1) J_{2/3}(\lambda z)\, d\lambda = \frac{1}{z_1^{1/3} z^{2/3}} \qquad z_1 < z$$
$$= 0 \qquad z_1 > z \tag{48}$$

$$\int_0^\infty \frac{\lambda^2}{\lambda^2 + \alpha^2} J_{-1/3}(\lambda z_1) J_{2/3}(\lambda z)\, d\lambda = \begin{cases} -\alpha I_{2/3}(\alpha z) K_{-1/3}(\alpha z_1) & z_1 > z > 0 \\ \alpha K_{2/3}(\alpha z) I_{-1/3}(\alpha z_1) & z > z_1 > 0 \end{cases} \tag{49}$$

The notation is that of Ref. 6. In order to have a stagnation point at the nose $x \to 0$ as $z \to \infty$, (cf. Eq. (26)). But, at $v = v_0$

$$x = 1 - (\tfrac{3}{2})^{1/3} A z_1 z^{2/3} \frac{2}{v_0} \sum_{m=1}^{\infty} (-)^m \left(\frac{m\pi}{v_0}\right) K_{2/3}\left(\frac{m\pi z}{v_0}\right) I_{1/3}\left(\frac{m\pi z_1}{v_0}\right) - (\tfrac{3}{2})^{1/3} A z_1 z^{2/3} \frac{1}{v_0 z^{2/3} z_1^{1/3}}$$

Now:

$$K_\nu(Z) \cong \sqrt{\frac{\pi}{2Z}}\, e^{-Z} + \cdots \tag{50}$$

so that as $z \to \infty$, we need

$$z_1 A = (\tfrac{3}{2})^{1/3} z_1^{1/3} v_0 \tag{51}$$

Summarizing, we have the following integral representation of the singular solution from (35) and (43)

$$y_s(z, v; z_1) = \left(\frac{2zz_1}{3}\right)^{1/3} v_0 \int_0^\infty \frac{\operatorname{sh} \lambda(v_0 - v)}{\operatorname{sh} \lambda v_0} J_{-1/3}(\lambda_s) J_{-1/3}(\lambda z_1) \lambda\, d\lambda$$
$$x_s(z, v; z_1) = 1 - z_1^{1/3} z^{2/3} v_0 \int_0^\infty \frac{\operatorname{ch} \lambda(v_0 - v)}{\operatorname{sh} \lambda v_0} J_{2/3}(\lambda z) J_{-1/3}(\lambda z_1) \lambda\, d\lambda \tag{52}$$

and a series representation of x on the surface of the wedge

$$x_s(z, v_0; z_1) = \begin{cases} -2z_1^{1/3} z^{2/3} \sum_{m=1}^{\infty} (-)^m \left(\frac{m\pi}{v_0}\right) K_{2/3}\left(\frac{m\pi z}{v_0}\right) I_{-1/3}\left(\frac{m\pi z_1}{v_0}\right) & z > z_1 > 0 \\ 1 + 2z_1^{1/3} z^{2/3} \sum_{m=1}^{\infty} (-)^m \left(\frac{m\pi}{v_0}\right) K_{-1/3}\left(\frac{m\pi z_1}{v_0}\right) I_{2/3}\left(\frac{m\pi z}{v_0}\right) & z_1 > z > 0 \end{cases} \tag{53}$$

The discontinuity in x at $z = z_1$ is only apparent. If we use

$$I_\nu(Z)K_{\nu+1}(Z) + K_\nu(Z)I_{\nu+1}(Z) = 1/Z \tag{54}$$

(Ref. 6, p. 80) and interpret the series

$$\sum_{n=1}^{\infty} (-)^n = -\tfrac{1}{2} \tag{55}$$

(53) is seen to be continuous at $z = z_1$.

Now the solution given by (52) will be discussed. This solution of (12) has a doublet singularity representing the free stream at $z = z_1$, has a stagnation point at the nose, satisfies the condition of tangency on the surface of the wedge and reaches the sonic velocity at the beginning of the corner. The wedge has unit length. However it does not satisfy the correct boundary conditions in the supersonic region, in particular the flow around the corner. Other non-singular solutions of (12) will have to be added to (52) in order to satisfy the conditions in the supersonic region ($u > 0$). These additional solutions must satisfy homogeneous conditions in the subsonic domain, and being non-singular are likely to be of small magnitude. In a sense, they represent upstream spreading of small disturbances in the supersonic region. Thus, in the subsonic domain ($w > 0$), the solution represented by (52) can be considered the main part of the actual solution.

It is also interesting to give another interpretation of (52) by seeing exactly what problem it does solve. (52) is the unique solution to a problem for (12) in the subsonic domain $w > 0$. The boundary conditions concerning the free stream, stagnation point and surface of the wedge are of course the same as before. In addition it is clear from (52) that on the sonic line,

$$x_s(0, v; z_1) = 1 \qquad 0 \leq v \leq v_0 \tag{56}$$

This result follows from the fact that solutions like Γ_I have $\partial y/\partial u = 0$ on the sonic line. Thus (52) is the solution to a problem where $x = 1$ on the sonic line. One boundary condition is given, for example on y, all around the elliptic domain and the singularity is also characterized. The solution is thus unique. Replacing the sonic line by $x = 1$ should be a good approximation, as far as the solution over the front part of the wedge is concerned. For low subsonic Mach numbers the supersonic zone is actually small and $x = 1$ is close to being correct. For large Mach numbers $x = 1$ differs more from the actual sonic line, traveling away from the wedge, but in this case disturbances or errors in the supersonic region have a weaker influence upstream.

The considerations above indicate that (52) should be good for finding the drag of the front part of the wedge and this is carried out in the next section.

5. Drag Coefficient of the Wedge. The drag coefficient is defined here in a standard way as

$$C_D = D/\tfrac{1}{2}\rho_\infty(a^* + u_{1\infty})^2 \quad \text{per unit width} \tag{57}$$

where

D = total drag of front part of wedge (upper half)

ρ_∞ = density at upstream infinity

The local pressure coefficient is also defined as

$$C_p = (p - p_\infty)/\tfrac{1}{2}\rho_\infty(a^* + u_{1\infty})^2 \tag{58}$$

where

p = local pressure

p_∞ = pressure at upstream infinity

Thus the drag coefficient referred to the free stream pressure is given by

$$C_D = \delta \int_0^1 C_p \, dx \qquad \text{for small } \delta \tag{59}$$

This drag coefficient applies most closely to the case where the wedge is followed by an infinite straight section. In other cases it is a measure of the nose drag based on an arbitrary reference level of pressure. For the linearized theory the pressure coefficient (58) can be approximated by

$$C_p = -2\frac{u_1 - u_{1\infty}}{a^*} = \frac{2}{\gamma + 1}\left(\frac{3}{2}\right)^{2/3}\{z^{2/3} - z_1^{2/3}\} \tag{60}$$

This may not actually be the best definition of C_p to use for practical work, but this point will not be discussed here (see Refs. 9 and 13). Now (59) becomes

$$C_D = -[2/(\gamma + 1)](\tfrac{3}{2})^{2/3}\delta\{C_1 + z_1^{2/3}\} \tag{61}$$

where

$$C_1 = -\int_0^1 z^{2/3}\, dx = \int_0^\infty z^{2/3}\,(\partial x/\partial z)\, dz \tag{62}$$

since $x = x(z)$ on the surface of the wedge. Using the series representation of x (53), C_1 can be evaluated

$$C_1 = 2z_1^{1/3}\sum_{m=1}^{\infty}(-)^m\left(\frac{m\pi}{v_0}\right)\left\{K_{1/3}\left(\frac{m\pi z_1}{v_0}\right)\int_0^{z_1} z^{2/3}\frac{\partial}{\partial z}\left[z^{2/3}I_{2/3}\left(\frac{m\pi z}{v_0}\right)\right]dz\right.$$
$$\left. - I_{-1/3}\left(\frac{m\pi z_1}{v_0}\right)\int_{z_1}^{\infty} z^{2/3}\frac{\partial}{\partial z}\left[z^{2/3}K_{2/3}\left(\frac{m\pi z}{v_0}\right)\right]dz\right\} \tag{63}$$

Integration by parts and the use of well-known formulae like

$$\frac{d}{d\sigma}[\sigma^{1/3}I_{-1/3}(\sigma)] = \sigma^{1/3}I_{2/3}(\sigma)$$
$$\frac{d}{d\sigma}[\sigma^{1/3}K_{-1/3}(\sigma)] = -\sigma^{1/3}K_{2/3}(\sigma) \tag{64}$$

permits the integrals in (63) to be evaluated as

$$\int_0^{z_1} z^{2/3}\frac{\partial}{\partial z}[z^{2/3}I_{2/3}(\alpha z)]\, dz = z_1^{4/3}I_{2/3}(\alpha z_1) - \frac{2}{3}\frac{z_1^{1/3}}{\alpha}I_{-1/3}(\alpha z_1) + \frac{2^{4/3}}{3\Gamma(\frac{4}{3})\alpha^{4/3}} \tag{65}$$

and

$$\int_{z_1}^{\infty} z^{2/3} \frac{\partial}{\partial z} [z^{2/3} K_{2/3}(\alpha z)]\, dz = - z_1^{4/3} K_{2/3}(\alpha z_1) - \frac{2}{3} \frac{z_1^{1/3}}{\alpha} K_{-1/3}(\alpha z_1) \tag{66}$$

Then upon using (54) and (55) we obtain

$$C_1 = -z_1^{2/3} + \frac{2^{7/3}}{3\Gamma(\frac{2}{3})} z_1^{1/3} \left(\frac{v_0}{\pi}\right)^{1/3} \sum_{m=1}^{\infty} (-)^m \frac{1}{m^{1/3}} K_{1/3}\left(\frac{m\pi z_1}{v_0}\right) \tag{67}$$

Thus, the drag coefficient as given by (61), becomes

$$C_D = \frac{2^{5/3}}{3^{1/3}\Gamma(\frac{2}{3})\pi^{1/3}} \frac{\delta^{4/3}}{(\gamma+1)^{2/3}} z_1^{1/3} \sum_{m=1}^{\infty} (-)^{m+1} \frac{1}{m^{1/3}} K_{1/3}\left[\frac{m\pi z_1}{(\gamma+1)\delta}\right] \tag{68}$$

an expression for the drag coefficient of the upper half wedge in terms of the wedge angle δ and the free stream Mach number $z_1 = \frac{2}{3}(1 - M_\infty^2)^{3/2}$. This series gives a finite value of drag C_D^* as the free stream Mach number approaches one ($z_1 \to 0$).

$$C_D^* = \frac{4}{(\gamma+1)^{1/3}} \frac{\delta^{5/3}}{3^{1/3}} \frac{\pi^{1/3}}{\Gamma^2(\frac{2}{3})} \frac{n(\frac{2}{3})}{\sin \frac{1}{3}\pi} \tag{69}$$

where

$$n(\tfrac{2}{3}) = \sum_{m=1}^{\infty} \frac{(-)^{m+1}}{m^{2/3}}$$

a series simply related to the Riemann zeta function tabulated in Ref. 8, p. 273. Guderley and Yoshihara (Ref. 10) have, by another method, computed the pressure distribution on the surface of the wedge. A numerical integration of their C_p curve (Ref. 9) gives

$$C_{D_{GY}}^* = 1.75\, \delta^{5/3}/(\gamma+1)^{1/3} \tag{70}$$

while an evaluation of (69) gives

$$C_D^* = 1.67\, \delta^{5/3}/(\gamma+1)^{1/3} \tag{71}$$

There is thus good agreement. As an example, for a wedge of 10° semi-apex angle $\delta = \pi/18$

$$C_D^* = .068$$

The series (68) converges rapidly if $\pi z_1/(\gamma+1)\delta$ is at all large because of the asymptotic behavior of $K_{1/3}$ (see Eq. (50)). For small values of the argument of $K_{1/3}$ it is more convenient to use a Taylor expansion about $M_\infty = 1$

$$C_D = C_D^* + (M_\infty - 1)\left(\frac{dC_D}{dM_\infty}\right)^* + \frac{(M_\infty - 1)^2}{2!}\left(\frac{d^2C_D}{dM_\infty^2}\right)^* + \cdots \tag{72}$$

By differentiating (68) and passing to the limit $z_1 = 0$ we obtain

$$(d\,C_D/dM_\infty)^* = [4/(\gamma+1)]\delta \tag{73}$$

and

$$(d^2C_D/dM_\infty^2)^* = [4/(\gamma + 1)]\delta \tag{74}$$

The slope of the drag curve at $M_\infty = 1$ is the same as can be obtained by other considerations (e.g. Ref. 9). The basic idea used in Ref. 9 for the derivation of (73) is that the velocity distribution over the wedge is exactly the same if the free stream Mach number is slightly below or slightly above one. When the

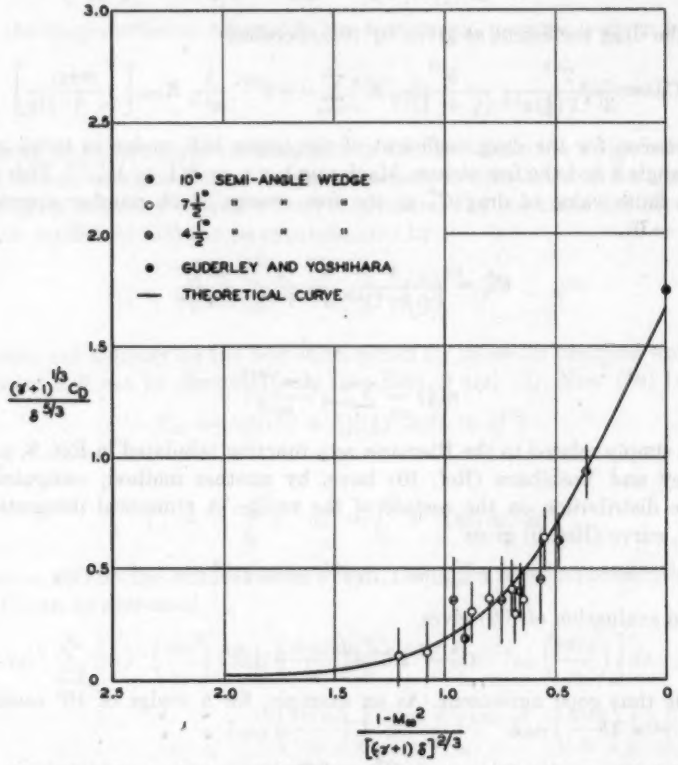

FIG. 3. Drag Coefficient of Upper Half Wedge As Function of Free Stream Mach Number

free stream Mach number is slightly greater than one there is an almost normal shock at a large distance in front of the wedge. Behind this shock the flow is again subsonic for a large distance on both sides of the axis of symmetry. The closer the Mach number is to one, the more nearly do the flow fields coincide. When considered in terms of local Mach number these facts imply

$$(\partial M/\partial M_\infty)^* = 0 \tag{75}$$

and when considered in terms of drag coefficient as defined here they give the result (73).

Curves of C_D versus M_∞ for various wedge angles have been computed from (67) and (72) and these are plotted in Fig. 3. Experimental points are also given. The method of obtaining these points is explained in Ref. 9.

6. Exact Boundary Value Problem. In this section some remarks will be made about the exact boundary value problem. This problem is involved mainly with the continuation of the flow into the supersonic region and the shock waves which occur. Thus it is not of too much interest for the present work but it becomes important when the flow over the rear half of a body is desired.

The first step in continuing the flow is satisfaction of the requirement that the flow is locally Prandtl-Meyer. This boundary condition, discussed before, is Eq. (28)

$$\text{on:}\quad v = v_0 - \tfrac{3}{4}u^{3/2} \qquad y(u, v) = 0 \tag{28}$$

Due to the linearity of the problem in the hodograph, correction solutions of (12) can be added to y_s of (52) in order to satisfy (28). These correction solutions are required to be non-singular and not to disturb the boundary conditions in the subsonic region. It is possible to derive relationship between y, y_u on the sonic line to insure that (28) is satisfied. Darboux' formula (Ref. 7) applies to (12) and gives the solution in the hyperbolic region in terms of y, y_u on the sonic line. Thus, within the domain of dependence of the Prandtl-Meyer flow, i.e. between the sonic line and the last characteristic from the corner to the sonic line (see Figs. 1 and 2) Darboux' formula is valid. Applying the formula to a point on the characteristic (28) and using the boundary condition we have

$$0 = \frac{\lambda^{2/3}}{\beta(\frac{1}{6}, \frac{1}{6})} \int_0^\lambda f(\alpha)\alpha^{-5/6}(\lambda - \alpha)^{-5/6}\, d\alpha + \frac{1}{\beta(\frac{5}{6}, \frac{5}{6})}\left(\frac{3}{4}\right)^{2/3} \int_0^\lambda g(\alpha)\alpha^{-1/6}(\lambda - \alpha)^{-1/6}\, d\alpha \tag{76}$$

where $\lambda = v_0 - v$, β = beta function, and

$$y(0, v) = f(\lambda) \qquad y_u(0, v) = g(\lambda)$$

It can be verified by substitution (cf. Ref. 7) that a solution of (76) is

$$f(\lambda) = \gamma_1 \int_0^\lambda \frac{g(\alpha)}{(\lambda - \alpha)^{1/3}}\, d\alpha \qquad \text{where} \qquad \gamma_1 = -\frac{3^{2/3}\Gamma^3(\frac{1}{3})}{4\pi^2} \tag{77}$$

and (77) can be inverted by Abel's formula as

$$g(\lambda) = \frac{1}{\pi\gamma_1} \sin\frac{\pi}{3}\frac{d}{d\lambda}\int_0^\lambda \frac{f(\alpha)}{(\lambda - \alpha)^{2/3}}\, d\alpha \tag{78}$$

(77) and (78) are the required relationships. A non-singular solution of (12) must be added to (52) to satisfy (77) on the sonic line. When this is done the last characteristic from the corner to the sonic line is determined. The flow can be continued across the last characteristic either by the method of characteristics

or by solving a characteristic initial value problem for Eq. (12). The characteristic initial value problem involves the value of y on the last characteristic and the condition (28).

Not too much more can be said at present about the flow pattern except that a shock wave must arise starting from the corner (see Refs. 2 and 3). Preliminary investigations indicate that either strong or weak shocks may occur. The weak shock has a supersonic velocity following it and is followed by a shock normal at the surface to achieve subsonic velocity. In order to complete the turn with an oblique shock the flow necessarily overexpands locally. In any case the shock pattern must fit in, as described before, by matching the value of (y, x) of the solution as continued in the supersonic zone to the value of (y, x) in the subsonic region, in one or two steps. The appropriate jump or jumps of (u, v) must satisfy the shock polar (29). It should also be remarked that the drag as computed from the entropy changes of the shocks (see Ref. 14) should agree with the pressure drag of the front portion of the wedge.

7. Conclusion. In this paper it has been shown how some importan parts of the flow field about a finite wedge can be found. It was also shown how a drag arises from the completely subsonic portion of the front part of the wedge and an explicit expression was found for this drag. In linearized subsonic flow the drag for the same problem is zero. The drag curve predicted from the transonic theory has the right general form and slope at $M_\infty = 1$ and agrees fairly well with the few experiments which have been performed. This type of a drag occurs also at supersonic speeds when there is a detached shock wave ahead of the wedge. This flow pattern has been recently computed by Vincenti (Ref. 13) using the same transonic flow approximations as were used here. The drag, as computed from two-dimensional theory is of the nature of an upper bound for the drag of wings of finite span and bodies. It should be noted that the drag coefficient predicted for the 10° wedge was quite large. Closing the body at the rear may reduce this drag somewhat, at least for lower Mach numbers.

A short discussion of the exact boundary value problem was also given and it was shown how shock waves can be regarded as fitting themselves into a basically subsonic flow pattern in order to satisfy the boundary conditions. For the future work it will be interesting to carry out the procedure sketched for the exact problem and to treat closed bodies.

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ON THE ROOTS OF ALGEBRAIC EQUATIONS

By Yudell L. Luke and Dolores Ufford

1. Introduction. A novel method for determining the roots of polynomials has been devised by S. N. Lin[1] [1], [2]. He defines an algorithm which gives rise to an iterative procedure so that when the process is convergent a given polynomial can be factored into two polynomials of lower degree. In actual practice, one of the latter polynomials is usually linear or quadratic. The algorithm is then applied to the other polynomial and the process is repeated so that the given polynomial is finally factored into a set of polynomials whose roots can be simply determined. Thus, to reduce an equation of degree n into n linear factors, the algorithm is applied $(n - 1)$ times; and to reduce an equation of degree $2n$ into n quadratic factors, the algorithm is also applied $(n - 1)$ times. It is the purpose of this paper to show how Lin's method can be extended so that a given polynomial can be completely separated into any number of sub-polynomials at the same time. Thus, if an equation has real roots, then all the roots can be simultaneously determined with little additional labor over that required to find a single root using Lin's approach.

2. Lin's method. To describe the extension, it is pertinent that Lin's algorithm be defined. Let the given algebraic equation of degree n

$$\sum_{i=0}^{n} A_i x^i = 0; \qquad A_n = 1 \tag{1}$$

be factored into two polynomials of degree m and $n - m$ as follows:

$$\left(\sum_{i=0}^{m} b_i x^i\right)\left(\sum_{i=0}^{n-m} a_i x^i\right) + \sum_{i=0}^{m-1} r_i x^i = 0; \qquad b_m = a_{n-m} = 1 \tag{2}$$

where the elements r_i define the remainder terms. For a given set of b_i's, the coefficients a_i and r_i are readily determined by synthetic division. We have,

$$a_{n-m-1} = A_{n-1} - b_{m-1} \tag{3}$$

$$a_{n-m-2} = A_{n-2} - b_{m-2} - b_{m-1}a_{n-m-1} \tag{4}$$

$$a_{n-m-3} = A_{n-3} - b_{m-3} - b_{m-1}a_{n-m-2} - b_{m-2}a_{n-m-1}, \text{ etc.} \tag{5}$$

$$r_0 = A_0 - a_0 b_0 \tag{6}$$

$$r_1 = A_1 - a_1 b_0 - a_0 b_1 \tag{7}$$

$$r_2 = A_2 - a_2 b_0 - a_1 b_1 - a_0 b_2 \tag{8}$$

and, in general

$$r_{m-1} = A_{m-1} - a_{m-1}b_0 - a_{m-2}b_1 - \cdots - a_1 b_{m-2} - a_0 b_{m-1} \tag{9}$$

[1] Numbers in brackets refer to bibliography at end of paper.

The coefficients a_i and b_i are evaluated by iteration. To do this, assume an initial set of values for the elements b_i. The coefficients a_i are then computed from (3), (4), (5), etc. If the set b_i were correctly chosen, the remainder terms r_i would vanish. If they are not zero, then a new approximation for the set b_i can be obtained from (6), (7), (8), etc., by using the a_i just computed and requiring the values of r_i in the latter equations to vanish. The set b_i so obtained is then employed in (3), (4), (5), etc., to obtain a second approximation for the a_i, and these in turn are employed to compute a new approximation for the b_i, etc. If, after repeating this process a sufficient number of times, the elements a_i and b_i remain unchanged so that the remainder terms r_i approach zero, the procedure converges and (1) is factored into two polynomials of lesser degree. If one of these polynomials is linear or quadratic, the root(s) are at hand. The other polynomial is then subjected to the above process, and in this manner (assuming convergence) all the roots of a given polynomial can be determined.

3. The extension of Lin's method. To extend Lin's method, we apply his algorithm to one of the two polynomials which are factors of the original polynomial (1). This gives rise to a set of equations similar to (3)–(5), etc., and (6)–(8), etc. If we require all remainder terms to vanish, and eliminate the set of coefficients common to all the equations, an algorithm is defined which enables one to factor the original polynomial into three polynomials of lesser degree.

Suppose then that Lin's algorithm is applied to the polynomial

$$\sum_{i=0}^{n-m} a_i x^i = 0 \tag{10}$$

Let this polynomial be factored into two polynomials of degree q and $n - m - q$ as follows:

$$\left(\sum_{i=0}^{q} C_i x^i\right)\left(\sum_{i=0}^{n-m-q} d_i x^i\right) + \sum_{i=0}^{q-1} \bar{r}_i x^i = 0; \qquad C_q = d_{n-m-q} = 1 \tag{11}$$

The coefficients a_i, C_i, d_i and $\bar{r}_i$ are related by the equations

$$d_{n-m-q-1} = a_{n-m-1} - C_{q-1} \tag{12}$$

$$d_{n-m-q-2} = a_{n-m-2} - C_{q-2} - C_{q-1}d_{n-m-q-1} \tag{13}$$

$$d_{n-m-q-3} = a_{n-m-3} - C_{q-3} - C_{q-1}d_{n-m-q-2} - C_{q-2}d_{n-m-q-1}, \text{ etc.} \tag{14}$$

$$\bar{r}_0 = a_0 - d_0 C_0 \tag{15}$$

$$\bar{r}_1 = a_1 - d_1 C_0 - d_0 C_1 \tag{16}$$

$$\bar{r}_2 = a_2 - d_2 C_0 - d_1 C_1 - d_0 C_2, \text{ etc.} \tag{17}$$

Suppose that $r_i = \bar{r}_i = 0$ for all i. To obtain an algorithm so that the given polynomial can be factored into three polynomials of lesser degree, we eliminate the set of coefficients a_i from (3)–(5), etc., (6)–(8), etc., (12)–(14), etc., and (15)–(17), etc. Thus, from (15),

$$C_0 = a_0/d_0 \tag{18}$$

and employing (3)–(5), etc., we can write

$$C_0 = F_0(A, b)/d_0 \tag{19}$$

where $F_0(A, b)$ is a function of the A_i's and b_i's. In general

$$C_i = F_i(A, b, C_{i-1}, d); \quad i = 1, 2, \cdots q-1 \tag{20}$$

Employing (6)–(8), etc., and (15)–(17), etc., it follows that

$$b_0 = A_0/C_0 d_0 \tag{21}$$

$$b_j = G_j(A, b_{j-1}, C, d); \quad j = 1, 2, \cdots m \tag{22}$$

Utilizing (12)–(14), etc., and (3)–(5), etc., we get

$$d_{n-m-q-1} = A_{n-1} - b_{m-1} - C_{q-1} \tag{23}$$

$$d_k = H_k(A, b, C, d_{k+1}); \qquad k = n - m - q - 2, \cdots, 1, 0 \tag{24}$$

The iteration procedure is now clear. Assume values for the set of coefficients C_i and b_i. Any values may be taken, although experience with practical examples indicates that the convergence is usually more rapid if the initial values of C_i and b_i are null. Employing (23) and (24), the set d_i is evaluated. Next the elements C_i are calculated using (18) and (20). Then (21) and (22) are used to determine the set b_i, etc. This process is repeated until the coefficients b_i, C_i and d_i do not change, and so if the procedure converges, the original polynomial is factored into three polynomials of lesser degree. The procedure to factor a given polynomial into any number of sub-polynomials obtains by extending the previous arguments.

4. The case of all linear factors. It is apparent that an attempt to write the general expressions to separate a polynomial into any number of sub-polynomials presents considerable algebraic difficulties. A more practical approach is to develop formulae for types of frequent occurrence in applied problems. We first consider the case of all linear factors. This situation is easily generalized. Let the factors of (1) be $(x + x_i)$, $i = 1, 2, \cdots n$. Then

$$x_{n-1} = \{A_{n-2} - A_{n-1}\sum x_i + \sum (x_i^2 + x_i x_j)\}/x_n \tag{25}$$

$$x_{n-2} = \{A_{n-3} - A_{n-2}\sum x_i + A_{n-1}\sum (x_i^2 + x_i x_j) - \sum (x_i^3 + x_i^2 x_j)\}/x_n x_{n-1} \tag{26}$$

$$x_{n-3} = \{A_{n-4} - A_{n-3}\sum x_i + A_{n-2}\sum (x_i^2 + x_i x_j) - A_{n-1}\sum (x_i^3 + x_i^2 x_j) + \sum (x_i^4 + x_i^3 x_j)\}/x_n x_{n-1} x_{n-2}, \text{ etc.} \tag{27}$$

$$x_1 = A_0/x_2 x_3 \cdots x_n, \tag{28}$$

$$x_n = A_{n-1} - \sum x_i \tag{29}$$

where the notation $\sum$ is used in the symmetric function sense, it being understood that in the formula for x_p,

$$\sum (x_i^k + x_i^{k-1} x_j) = \sum_{\substack{i,j=1 \\ i \neq j}}^{p-1} (x_i^k + x_i^{k-1} x_j) \tag{30}$$

Example I: Consider the quintic equation

$$x^5 + 33.7x^4 + 313.16x^3 + 804.98x^2 + 437.4x + 36 = 0.$$

Employing (25), (26), $\cdots$ (29), the successive approximations are listed below. Note that as in Lin's method a first approximation is obtained by taking $x_4 = x_3 = x_2 = x_1 = 0$.

Approximation	x_4	x_3	x_2	x_1	x_0
0	0	0	0	0	33.7
1	9.29	2.57	0.54	0.082	21.22
2	10.09	2.90	0.60	0.097	20.01
3	10.13	2.96	0.60	0.099	19.91
4	10.10	2.98	0.60	0.100	20.01
5	10.02	2.99	0.60	0.100	19.99
6	10.02	3.00	0.60	0.100	19.98

The exact roots are -0.1, -0.6, -3.0, -10.0, and -20.0.

Suppose that the quintic is separated into a linear and a quartic equation. Utilizing (3), (4), $\cdots$ (9), the approximations are given in the following table.

Approximation	$b_0 = A_4/a_0$	$a_1 = A_4 - b_0$	$A_2 = A_3 - a_1 b_0$	$a_2 = A_2 - a_1 b_0$	$a_3 = A_1 - a_2 b_0$
0	0	33.7	313.16	804.98	437.4
1	0.082	33.62	310.40	779.53	373.48
2	0.096	33.60	309.93	775.23	362.98
3	0.099	33.60	309.83	774.31	360.74
4	0.100	33.60	309.80	774.00	360.00

Note that when Lin's method is employed one linear root is determined after four approximations. To factor the quartic into a pair of quadratics requires five approximations; but utilizing the approach of this paper, all the roots are known after six approximations. Thus the technique outlined here offers considerable advantage over that due to Lin.

5. Convergence. The question of convergence of Lin's method and the extension presented here has not been fully answered, although progress in this direction has been made by Friedman [3]. He has developed a modification of Lin's method, and has shown that if Lin's method converges, the technique of his paper converges more rapidly. If, starting with a trial factor near the true one, the factor obtained by iteration is closer to the true factor, then the latter is said to be "stable." On the other hand, if the resulting factor obtained by iteration is farther away from the true factor, then the latter is said to be "unstable." In general, it is shown that the roots smallest in absolute value are "stable." In Friedman's procedure, only one factor is found at a time.

From the examples presented here and numerous others encountered in the computational laboratory of the Midwest Research Institute, the method of this

paper converges quite rapidly if all the roots of the polynomial lie in the left half plane, and if the roots are well separated. These qualitative results agree with those deduced by Lin and Friedman. When using Lin's method or its extension, the concept of root separation is somewhat elastic. For example, if two linear factors are near each other, but well separated from a pair of complex roots which are in turn well separated from another pair of complex roots, then separation of the given polynomial by quadratics will give rapid convergence. However, in the case just cited, an attempt to separate the given equation using linear factors fails.

6. The separation of a sextic into three quadratics. Suppose that it is desired to resolve a 6th degree polynomial into three quadratics. Let the quadratics be defined by $x^2 + b_1x + b_0$, $x^2 + C_1x + C_0$ and $x^2 + d_1x + d_0$. Then the equations to iterate are

$$C_0 = (A_2 - b_1A_3 - b_0A_4 + b_1^2A_4 + 2b_1b_0A_5 - b_1^3A_5 + b_0^2 - 3b_1^2b_0 + b_1^4)/d_0 \quad (31)$$

$$C_1 = (A_3 - b_1A_4 - b_0A_5 + b_1^2A_5 + 2b_1b_0 - b_1^3 - C_0d_1)/d_0 \quad (32)$$

$$b_0 = A_6/C_0d_0 \quad (33)$$

$$b_1 = \{A_1 - b_0(C_1d_0 + C_0d_1)\}/C_0d_0 \quad (34)$$

$$d_1 = A_5 - b_1 - C_1 \quad (35)$$

$$d_0 = A_4 - b_0 - b_1A_5 + b_1^2 - C_0 - C_1d_1 \quad (36)$$

When using these equations, the iteration can be started by taking $C_0 = C_1 = b_0 = b_1 = 0$.

Example II: Let the given equation be

$$x^6 - 2.2x^5 + 105.2x^4 - 140.6x^3 + 392.1x^2 + 273.9x + 65.1 = 0.$$

Employing (31)–(36), the approximations obtained are tabulated below.

Approximations	C_0	C_1	b_0	b_1	d_1	d_0
0	0	0	0	0	−2.2	105.2
1	3.73	−1.26	0.166	0.758	−1.70	101.41
2	5.35	−2.09	0.120	0.553	−0.659	99.87
3	4.90	−1.96	0.133	0.614	−0.852	100.22
4	5.03	−2.01	0.129	0.596	−0.785	100.13
5	4.99	−2.00	0.130	0.601	−0.804	100.16
6	5.00	−2.00	0.130	0.600	−0.800	100.15

This example is used by Lin who separates the sextic into a quadratic and a quartic. After six approximations the quadratic $x^2 + 0.600x + 0.13$ is obtained. This is the same number of approximations used above, and so again the present method affords a saving of time and labor.

7. Utility of the present method. We now present an example which sheds further light on the utility of the present method.

Example III: Consider the sextic equation

$$x^6 + 256.79x^5 + 152.01x^4 + 258.43x^3 + 88.37x^2 + 4.52x + 0.0443 = 0.$$

Suppose that one tries to separate this into three quadratics. The approximations found are given in the table below.

Approximations	C_2	C_1	b_0	b_1	d_1	d_0
0	0	0	0	0	256.79	152.01
1	0.581	1.634	0.00050	0.0489	256.16	−279.75
2	−0.274	−1.114	0.00058	0.0572	257.01	423.85
3	0.176	0.461	0.00059	0.0587	256.28	18.72
4	3.963	−41.41	0.00060	0.0590	298.15	12,480.24

Note that one quadratic converges even though the coefficients which define the other two quadratics do not converge. Further, the quadratic which converges defines the pair of roots smallest in magnitude. If the convergent quadratic is removed from the sextic, the above results indicate most likely that the smallest and largest roots in absolute value of the quartic so obtained are real, and the magnitude of the intermediate roots are either equal, in which case they may be complex, or nearly equal. This suggests that the quartic be separated into a quadratic and two linear factors. Let the factors of (1) be designated by $(x + x_1)$, $(x^2 + b_1x + b_0)$ and $(x + x_2)$. Then the equations to iterate are

$$b_0 = (A_1 - x_1A_2 + x_1^2A_3 - x_1^3)/x_2 \tag{37}$$

$$b_1 = (A_2 - x_1A_3 + x_1^2 - b_0)/x_2 \tag{38}$$

$$x_1 = A_0/b_0x_2 \quad (39); \qquad x_2 = A_3 - x_1 - b_1 \quad (40)$$

Example IV: The quartic obtained by removing the known quadratic from the sextic in Example III is

$$x^4 + 256.73x^3 + 136.86x^2 + 250.21x + 73.53 = 0.$$

Utilizing (37)–(40) the computed approximations are given in the following table.

Approximations	b_0	b_1	x_1	x_2
0	0	0	0	256.73
1	0.975	0.529	0.294	255.91
2	0.907	0.237	0.317	256.18
3	0.908	0.213	0.316	256.20
4	0.908	0.214	0.316	256.20

Thus, Examples III and IV show that though the attempt to separate the given sextic into three quadratic factors yields but one convergent quadratic, sufficient

information is obtained so that the remaining roots of the sextic can be quickly computed. Applying Lin's algorithm to the sextic, one could obtain the quartic given above. If the quartic is separated into two quadratics, the iteration process does not converge and so one would attempt to separate the quartic using a linear and a cubic. However, the present method avoids these difficulties.

In the derivation of (37)–(40), (1) is first divided by $x + x_1$. The quotient so obtained is divided by $x^2 + b_1x + b_0$ and the new quotient is divided by $x + x_2$. In this manner, it turns out that when the method converges, the root smallest in magnitude is given by the factor $x + x_1$. The intermediate roots are defined by the quadratic $x^2 + b_1x + b_0$, and the factor $x + x_2$ yields the root largest in absolute value. Thus, these equations will not converge if the magnitude of the two smallest or largest roots are equal or nearly equal. If the sequence of division is $x^2 + b_1x + b_0$, $x + x_1$ and $x + x_2$, then a set of equations is obtained which differs from (37)–(40) in arrangement only. If the iteration converges, then the quadratic defines the roots smallest in magnitude and the largest roots are given by the two linear factors. However, there is no point in arranging the equations for this situation, since it is easier to separate the quartic into two quadratics.

To further illustrate the utility of the present method, consider a sextic equation. Suppose it is known that both real and complex roots are present. Assume that the real roots are well separated from each other and are also well separated from the complex roots. Let the given equation be separated into linear factors. Suppose that in the iteration, two roots converge and the others do not. The convergent roots are those smallest in absolute value. The next largest roots must be complex. The largest roots are either real or complex. Thus, the quartic defining the four largest roots should be separated into two quadratics. The exact nature of the largest roots are easily determined. Simply replace x by its reciprocal in (1) and redefine the coefficients A_i so that the coefficient of x^6 is unity. If the iteration process converges, then the largest roots are real; if not, then they are imaginary.

8. Conclusions. The extension of Lin's method presented here offers considerable computational advantages over the original and other methods for determining the roots of polynomials. If the method completely converges, then all the roots are simultaneously evaluated, thus affording a substantial saving in time and labor; but even if the method is only partially convergent, so that some of the roots are computed, sufficient information can be extracted to readily compute the remaining roots. Only a limited number of examples giving formulae for separation of a polynomial into sub-polynomials has been presented. A more extensive set of formulae has been developed to cover a number of cases depending on the nature of the roots, and will be made available to the reader upon request. The formulae are somewhat cumbersome for large degree polynomials. The methods of this paper are easily adapted to digital machine calculators if the degree of the polynomial is not larger than six. If the degree of the given polynomial is larger than six, it is probably best to use Lin's method to separate the polynomial into sub-polynomials of degree equal to or less than

six, and then apply the techniques outlined here. However, if punched card facilities are at hand, one should be able to apply the procedure developed in this note directly to polynomials of degree somewhat larger than six.

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SOME ROOTS OF AN EQUATION INVOLVING BESSEL FUNCTIONS

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In certain boundary value problems it is desirable to know the zeros of the following expression in which k is a real parameter:[1]

$$F(k, x) = J_0(x)N_1(kx) - N_0(x)J_1(kx). \tag{1}$$

In the above equation, N_0 and N_1 are the Bessel functions of the second kind as discussed and tabulated in Jahnke and Emde, Tables of Functions. They are the same as the functions denoted by Y_0 and Y_1 in Watson, Theory of Bessel Functions, second edition, page 64.

This note gives tables of the smallest zero $x_0(k)$ of (1) for $k = 1.0\ (0.1)\ 2.1$, $1.0\ (1)\ 10, 20$. In addition, by the simple transformation

$$k' = 1/k, \qquad x' = kx \tag{2}$$

we are able to determine the smallest zeros of the expression

$$G(k', x') = N_1(x')J_0(k'x') - J_1(x')N_0(k'x') \tag{3}$$

To find the smallest zero of (1) we introduce the variable

$$y(k) = (2/\pi)(k - 1)x_0 .$$

It can be shown that

$$\lim_{k \to 1} y = 1$$

An estimate of $x_0(1.1)$ was made by assuming $y(1.1) = 1$. Three values $F(x_1)$, $F(x_2)$, $F(x_3)$ were found using the British Association Mathematical Tables of Bessel Functions[2] such that

$$x_3 = x_2 + 0.1 = x_1 + 0.2$$

and

$$F(x_1) \leq 0 \leq F(x_3).$$

A parabola was passed through these points and the appropriate root of the quadratic so formed was taken as x_0. From this the value of $y(1.1)$ was found

[1] Other tables of roots of Eq. 1 are the following: Dinnik, Izvestia, Vol. 11, No. 2, 1915, pp. 23–24, $J_1(x)N_0(kx) - N_1(x)J_0(kx)$ for $k^2 = .2(.2)1$, 1st zero [2D].

Dinnik, ASME Trans. Vol. 51, 1929, APM-51-11 p. 109, $J_0(x)N_1(xe^{m/2}) - N_0(x)J_1(xe^{m/2})$ for $e^m = k^2 = .01, .2(.2)1$, 1st zero, [2–3D].

Carsten and McKerrow—unpublished manuscript, $J_0(x)N_1(kx) - N_0(x)J_1(kx)$ for $k = (1.1)^{1/2}$, 1.06 (.02) 1.2 (.5)1.5, 2(1) 5, [3, 4D].

Rendulic, Wasserwirtschaft u. Tech. Nr. 25–26, 1935, p. 270, $J_0(x)N_1(kx) - N_0(x)J_1(kx)$ $k = 0(.1)1.00, 1.0(1)7.0, 6.25$ 1st 6 roots [3, 4D].

Graphical material is available in Jahnke and Emde: Tables of Functions, 3rd. ed. p, 207, 208. Material taken from Rendulic, loc. cit.

Durant, N. J., Phil. Mag., Vol. 36, 1945, pp. 572–577, $J_0(\beta\theta)N_1(\theta) - N_0(\beta\theta)J_1(\theta)$ for $1.00 \leq \beta^2 \leq 40$, 12 entries first root [4D].

[2] British Association for the Advancement of Science, Mathematical Tables Vol. VI, Bessel Functions, Part 1, Cambridge University Press, 1937.

which enabled $y(1.2)$ to be estimated. The same procedure of taking the points x_1, x_2 and x_3 and finally a quadratic in x was used for all values of x computed, the interval $x_3 - x_2 = x_2 - x_1$ being made as small as possible, generally 0.01 or 0.005.

When $J_0(x) = 0$, it is necessary that $J_1(kx) = 0$, so that another value of $x_0(k)$ was found without computation. Similarly, another value was found where $N_0(x) = 0$. Table I shows the roots found, and the corresponding values of y. The values of $x_0(k)$ given were rounded off from their computed values and certain values were checked using an interval in x of .001 or less, x_0 being found by linear interpolation. It is felt that the values of x_0 are accurate to the number of places given.

An intermediate table for $y(k)$, $k = 1.0(.01)2.0$, was found by interpolation, using Everett's central-difference interpolation formula.

Table II gives $x_0(k)$, $k = 1.00$ (.01) 2.00, 2.1, 3(1)10, 20; and Table III gives $x_0'(k)$,$1/k' = 1.00(.01)$ 2.00, 2.1, 3(1)10, 20, roots of Eg. (3).

Tables II and III were made by Miss R. A. Weiss and Miss C. L. Froelich, who also checked Table I. Many thanks to them and to Mr. R. W. Hamming for his suggestions and help.

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TABLE I

k	x_0	y
1.0	∞	+1.0000
1.1	+15.406	.9808
1.2	7.567	.9634
1.25	6.003	.9553
1.3	4.9617	.9476
1.4	3.6646	.9332
1.5	2.8899	.9199
1.59334	2.40483	.90838
1.6	2.3761	.9076
1.7	2.0110	.8962
1.8	1.7388	.8856
1.9	1.5282	.8756
2.0	1.3608	.8663
2.1	1.22455	.8575
2.45882	.89358	.8299
3.0	.62556	.7965
4.0	.39346	.7514
5.0	.28236	.7190
6.0	.21807	.6942
7.0	.17651	.6742
8.0	.14760	.6578
9.0	.12642	.6439
10.0	.11027	.6318
20.0	.04652	.5627

TABLE II

First root of $J_0(x_0)N_1(kx_0) - N_0(x_0)J_1(kx_0)$

k	x_0	k	x_0	k	x_0
1.00	∞	1.37	3.980	1.74	1.893
1.01	156.8	1.38	3.869	1.75	1.866
1.02	78.23	1.39	3.764	1.76	1.839
1.03	52.05	1.40	3.665	1.77	1.813
1.04	38.96	1.41	3.570	1.78	1.788
1.05	31.10	1.42	3.480	1.79	1.763
1.06	25.87	1.43	3.394	1.80	1.739
1.07	22.13	1.44	3.312	1.81	1.716
1.08	19.33	1.45	3.234	1.82	1.692
1.09	17.15	1.46	3.159	1.83	1.670
1.10	15.41	1.47	3.087	1.84	1.648
1.11	13.98	1.48	3.019	1.85	1.627
1.12	12.79	1.49	2.953	1.86	1.606
1.13	11.79	1.50	2.890	1.87	1.586
1.14	10.924	1.51	2.829	1.88	1.566
1.15	10.178	1.52	2.771	1.89	1.547
1.16	9.525	1.53	2.715	1.90	1.528
1.17	8.948	1.54	2.661	1.91	1.510
1.18	8.436	1.55	2.609	1.92	1.492
1.19	7.979	1.56	2.559	1.93	1.474
1.20	7.567	1.57	2.511	1.94	1.457
1.21	7.194	1.58	2.465	1.95	1.440
1.22	6.855	1.59	2.420	1.96	1.423
1.23	6.546	1.60	2.376	1.97	1.407
1.24	6.263	1.61	2.334	1.98	1.391
1.25	6.003	1.62	2.294	1.99	1.376
1.26	5.762	1.63	2.254	2.00	1.361
1.27	5.540	1.64	2.216	2.10	1.225
1.28	5.333	1.65	2.179	3.00	.6256
1.29	5.141	1.66	2.144	4.00	.3935
1.30	4.962	1.67	2.109	5.00	.2824
1.31	4.794	1.68	2.075	6.00	.2181
1.32	4.637	1.69	2.043	7.00	.1765
1.33	4.489	1.70	2.011	8.00	.1476
1.34	4.351	1.71	1.980	9.00	.1264
1.35	4.220	1.72	1.950	10.00	.1103
1.36	4.096	1.73	1.921	20.00	.04652

TABLE III

First root of $N_1(x_0')J_0(k'x_0') - J_1(x_0')N_0(k'x_0')$

k'	x_0'	k'	x_0'	k'	x_0'
.05	.9304	.5780	3.323	.7353	5.571
.1000	1.103	.5814	3.354	.7407	5.697
.1111	1.138	.5848	3.386	.7463	5.830
.1250	1.181	.5882	3.419	.7519	5.971
.1429	1.236	.5917	3.452	.7576	6.121
.1667	1.309	.5952	3.486	.7634	6.280
.2000	1.412	.5988	3.522	.7692	6.450
.2500	1.574	.6024	3.558	.7752	6.632
.3333	1.877	.6061	3.595	.7813	6.826
.4762	2.573	.6098	3.634	.7874	7.036
.5000	2.722	.6135	3.674	.7937	7.260
.5025	2.738	.6173	3.716	.8000	7.503
.5051	2.754	.6211	3.758	.8065	7.766
.5076	2.771	.6250	3.802	.8130	8.052
.5102	2.789	.6289	3.848	.8197	8.363
.5128	2.808	.6329	3.895	.8265	8.705
.5155	2.827	.6369	3.942	.8333	9.080
.5181	2.846	.6410	3.992	.8403	9.495
.5208	2.865	.6452	4.044	.8475	9.954
.5236	2.884	.6494	4.098	.8547	10.469
.5263	2.903	.6536	4.154	.8621	11.05
.5291	2.923	.6579	4.212	.8696	11.70
.5319	2.944	.6623	4.272	.8772	12.45
.5348	2.965	.6667	4.335	.8850	13.32
.5376	2.987	.6711	4.400	.8929	14.32
.5405	3.009	.6757	4.468	.9009	15.52
.5435	3.032	.6803	4.538	.9091	16.95
.5465	3.055	.6849	4.612	.9174	18.69
.5495	3.079	.6897	4.689	.9259	20.88
.5525	3.105	.6944	4.769	.9346	23.68
.5556	3.130	.6993	4.853	.9434	27.42
.5587	3.156	.7042	4.942	.9524	32.66
.5618	3.182	.7092	5.034	.9615	40.52
.5650	3.209	.7143	5.131	.9709	53.61
.5682	3.237	.7194	5.232	.9804	79.79
.5714	3.265	.7246	5.339	.9901	158.3
.5747	3.294	.7299	5.453	1.00	∞

BOUNDARY LAYER IN CONVERGENT FLOW BETWEEN SPIRAL WALLS

By G. Kuerti*

1. Introduction. The usual boundary layer theory is based on a *simplification* of the Navier-Stokes equations. Certain well-known assumptions, based on actual observation, are introduced *in addition*, as it were, to the dynamical equations and the boundary conditions (Prandtl, 1904).

It has long been recognized that a stricter mathematical interpretation of boundary layer theory is desirable. Kármán [1] pointed out that boundary layer theory should be the *asymptotic* integration theory of the Navier-Stokes equation for $R \to \infty$, and Mises [2], in rederiving the boundary layer equations, gave conditions under which the Navier-Stokes equations (*without* a complete set of boundary conditions) admit asymptotic integrals corresponding to boundary layer flow. But the establishment of asymptotic solutions in the case of a *complete* viscous flow problem presents formidable difficulties. If the two-dimensional, steady flow problem is formulated in terms of the stream function ψ, the corresponding differential equation

$$\nu\nabla^2\nabla^2\psi = \psi_y\nabla^2\psi_x - \psi_x\nabla^2\psi_y \tag{1.1}$$

today hardly admits a systematic search for asymptotic solutions ($\nu \to 0$) along reasonably general boundaries.

There exist several important investigations of the boundary layer phenomenon with simpler equations. Wasow [4] studied a certain linear homogeneous *ordinary* differential equation, Mises [5] studied the equation $y' = f(x, y) + \nu g(x, y)y''$, and *partial* differential equations of second order were also investigated (Wasow [6], Levinson [7]). For Eq. (1.1), however, only one particular solution is known to converge towards boundary layer flow when $\nu \to 0$, viz. Hamel's convergent (sink) flow between *radial* walls [8]. The solution of the corresponding boundary layer problem, which already appears in [3], p. 266, can in this case be obtained as the asymptotic form of Hamel's solution [9].

The radial flow is a particular case in a family of solutions of (1.1) discussed in [8] the streamlines of which are logarithmic spirals. A sink flow between *spiral* walls belongs to this family, and the question arises whether the solution of the corresponding boundary layer problem can again be obtained as the asymptotic form of Hamel's solution. It will be shown that this is true for any width of channel and any degree of obliqueness of the spiral streamline, with the obvious exception that the spiral degenerates into a circle.

2. Reduction to an ordinary differential equation. *Solution in the radial case.* In the first part of [8], Hamel determines the most general solution of (1.1),

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when the family of the streamlines is required to be one of the two components of an isometric network. This calls for

$$\psi = f(\varphi) \quad \text{and} \quad \nabla^2\varphi = 0. \tag{2.1}$$

The function f must satisfy

$$f''f'b = \nu[f'''' + f''(a^2 + b^2) + 2af'''], \tag{2.2}$$

with constant a and b, while φ has the form

$$\varphi = -[2/(a^2 + b^2)](a \log r + b\vartheta) + \varphi_0, \tag{2.3}$$

r and ϑ being polar coordinates in the plane of flow. Obviously, φ_0 can be omitted in (2.3), and the family $\varphi = \text{const}$ remains the same when a and b are replaced by ca and cb. We then may normalize φ by prescribing

$$2b/(a^2 + b^2) = -1, \tag{2.4}$$

and write (2.3) in the form $r = \exp[(-b/a)(\vartheta - \varphi)]$. The mark φ is now equal to the ϑ-value at which the spiral meets the unit circle, and ϑ becomes therefore the independent variable in (2.2). The spiral angle γ is $\operatorname{arccot}(-b/a)$, and we may assume $0 \leqq \gamma \leqq \pi/2$. In view of (2.4), we put

$$a = \sin 2\gamma, \qquad b = -2\cos^2\gamma, \qquad (0 \leqq a \leqq 1, -2 \leqq b \leqq 0); \tag{2.5}$$

for circular flow $\gamma = \pi/2$, $a = 0$, $b = 0$; for radial flow $\gamma = 0$, $a = 0$, $b = -2$. From (2.1), (2.3), and (2.4), the velocity $\bar{q}$ is obtained as

$$q_r = \frac{1}{r}\frac{\partial\psi}{\partial\vartheta} = \frac{f'}{r}, \qquad q_\vartheta = -\frac{\partial\psi}{\partial r} = \frac{a}{-b}\frac{f'}{r}, \qquad q = \left(\frac{2}{-b}\right)^{1/2}\frac{f'}{r}. \tag{2.6}$$

According to (2.6), the flow moves toward a sink at 0 when $f' < 0$, and $|f'|$ is the velocity along a circular arc with radius $(-2/b)^{1/2}$. We shall only consider arcs such that f' is negative and $-f'$ assumes a proper maximum $-f'_0$ at an inner point. For the existence of such arcs see next section. Let φ_1 and φ_2 be two spiral streamlines passing through the extremities of the arc and introduce the dimensionless velocity

$$z = f'/f'_0, \tag{2.7}$$

which equals $q/q_{\max}$ along any arc with center 0, passing between φ_1 and φ_2 (cf. Fig. 1, where $f'(\varphi_1) = f'(\varphi_2) = 0$ and $f'(0) = f'_0$ is assumed). On choosing some characteristic radius r^* and associated $q^*_{\max}$, the Reynolds number of the flow, R, can be defined by

$$\nu R = q^*_{\max}\cdot r^* = -f'_0\cdot(-2/b)^{1/2} = -f'_0/m, \tag{2.8}$$

where we have written for brevity

$$b = -2m^2, \qquad (m = \cos\gamma). \tag{2.9}$$

Eq. (2.2), integrated once, reads in this notation

$$z'' + 2az' + 4m^2z - m^3Rz^2 + C = 0 \tag{2.10}$$

where C is an integration constant, $a = 2m(1 - m^2)^{1/2}$, and $m > 0$ is assumed from now on.

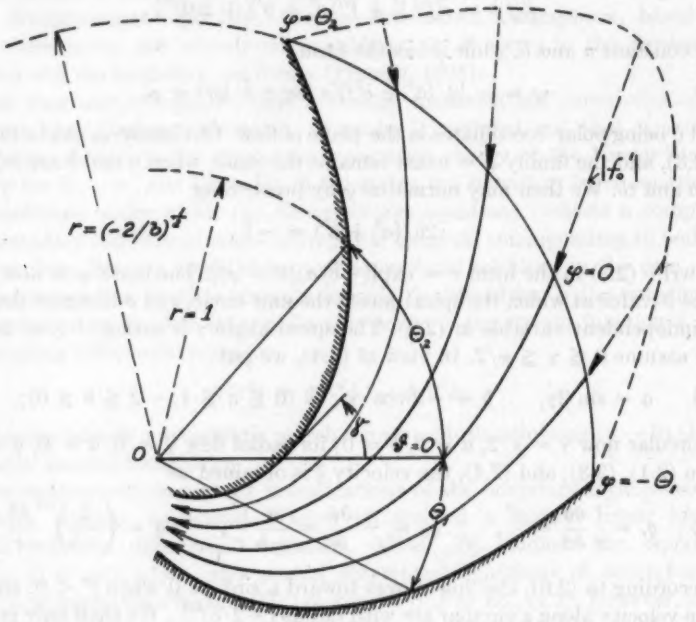

FIGURE 1

For radial flow ($m = 1, a = 0$), the general integral of (2.10) can be expressed in terms of elliptic functions. For the initial conditions $\{z = 1, z' = 0 \text{ at } \vartheta = 0\}$ one obtains, when $C > -2 + R/3$ for $0 < \mathrm{R} \leqq 3$ and $\mathrm{C} > -4 + \mathrm{R}$ for $R \geqq 3$

$$z = \frac{1 - \xi_1 k^2 sn^2(A\vartheta;k)}{1 - k^2 sn^2(A\vartheta;k)}. \tag{2.11}$$

Here $\xi_{1,2} = \frac{1}{2}[6R^{-1} - 1 \pm (36R^{-2} + 12R^{-1} - 3 + 12R^{-1}\,C)^{1/2}]$, $\xi_1 > 0 > \xi_2$, $A = [(\xi_1 - \xi_2)R/6]^{1/2}$, and the modulus $k = [(1 - \xi_2)/(\xi_1 - \xi_2)]^{1/2}$. The conditions given for C are necessary and sufficient for $z(\vartheta)$ to drop monotonically to zero when $\vartheta \to \pm\,\theta$. For $R \to \infty$, $\xi_1 \to 1$, $\xi_2 \to -2$.

In the general case $m < 1$, Eq. (2.10) can only be solved by elliptic functions when a certain relation[1] obtains between the coefficients, which cannot be valid, however, for the limiting process later considered. Such solutions have been used in a numerical study of spiral channel flow by Olsson and Faxén [10].

[1] viz. $(2^4 \cdot 3^2/5^4)a^4 - 2^2m^4 = m^2RC$ (Picard and Mittag-Leffler, quoted in [10]).

3. Solutions satisfying the boundary conditions $z = 0$. Since ϑ does not explicitly occur in (2.10), the point $\vartheta = 0$ may be chosen as abscissa to the maximum value of z (unity). A solution $z(\vartheta)$ of the initial value problem $\{z = 1, z' = 0 \text{ at } \vartheta = 0\}$ represents a channel flow, if (a) two positive numbers Θ_1 and Θ_2 exist such that $z = 0$ for $\vartheta = -\Theta_1$, $\vartheta = \Theta_2$, but $0 < z \leqq 1$ for $-\Theta_1 < \vartheta < \Theta_2$, and if (b) the channel width $\Theta = \Theta_1 + \Theta_2 < 2\pi$.

Obviously, $z''(0) < 0$ is a necessary condition, that is,

$$z''(0) = m^3R - 4m^2 - C < 0 \tag{3.1}$$

determines a lower bound for the admissible values of C, but it is also a sufficient condition as will be shown now.[2]

Eq. (2.10) is equivalent to the first order system

$$\begin{aligned} dz/d\vartheta &= p, \\ dp/d\vartheta &= -2ap - 4m^2z + m^3Rz^2 - C \qquad \equiv h(p, z), \end{aligned} \tag{3.2}$$

the r.h. sides of which are *polynomials* in p and z. It is well known that there exists in this case one and only one characteristic through any point of the phase plane z, p, with the exception of the critical points for which both r.h. sides of (3.2) vanish simultaneously. The only (finite) critical points in our case are the roots of $h(0, z) = 0$, viz.

$$\alpha_\nu = (2/mR)[1 - (-1)^\nu(1 + RC/4m)^{1/2}], \qquad \nu = 1, 2. \tag{3.3}$$

Since finally $R \to \infty$, we consider (3.3) for $R > 4/m$, say; C is then *positive*, cf. (3.1), and α_ν certainly real. One confirms easily that α_1 is a saddle point and α_2 a focal point; also $\alpha_2 < 0$, and from (3.1)

$$\alpha_1 > 1. \tag{3.4}$$

According to (3.2), vertical or horizontal characteristic elements can *only* occur along $p = 0$ or $h = 0$, respectively. The distribution of signs of $dp/d\vartheta$ and $dz/d\vartheta$ is indicated in Fig. 2, where the vertex of $h = 0$ has been denoted by A, the point $z = h = 0$ by B, and the point $z = 1, p = 0$ by Q. The half-characteristics S_1 and S_2 arrive at and depart from the saddle at slopes $-a \pm [a^2 + m^3R(\alpha_1 - \alpha_2)]^{1/2}$, coming from and going to the "*inside*" of $h = 0$. Along the parabolic arc $\alpha_1 A \alpha_2$ all possible crossings of characteristics pass *from right to left*, hence S_2 can only cross this arc between A and α_2; it either meets the p-axis between 0 and B or crosses the parabola between A and B; in the latter case, S_2 has a minimum at the crossing and, since it cannot re-cross between A and α_2, it must meet the p-axis somewhere between the ordinates p_B and p_A.—To discuss S_1, we write $\tau = -\vartheta$ and reverse the arrow; then $dp/d\tau > 0$ and $dz/d\vartheta < 0$ on S_1 in a certain neighborhood of α_1. Thus at any large negative τ, the phase point has left α_1 "along the tangent" and acquired some posi-

[2] This is intuitively clear if one interprets (2.10) as a damped non-linear oscillation with constant impressed force $-C$; condition (3.1) then stipulates that at the instant of release the total force be a restoring one.

tive value of p, hence negative $dz/d\tau$; the absolute value of that quantity can serve as a lower bound for all "later" $|\,dz/d\tau\,|$-values in the first quadrant. S_1 must therefore meet the p-axis at some finite τ-value.[3]

Let the characteristic arc that contains Q and has $z \geqq 0$ consist of arcs P_i, $i = 1, 2$ (Fig. 2). Then *either* P_i is penned between the z-axis and S_i and *there-*

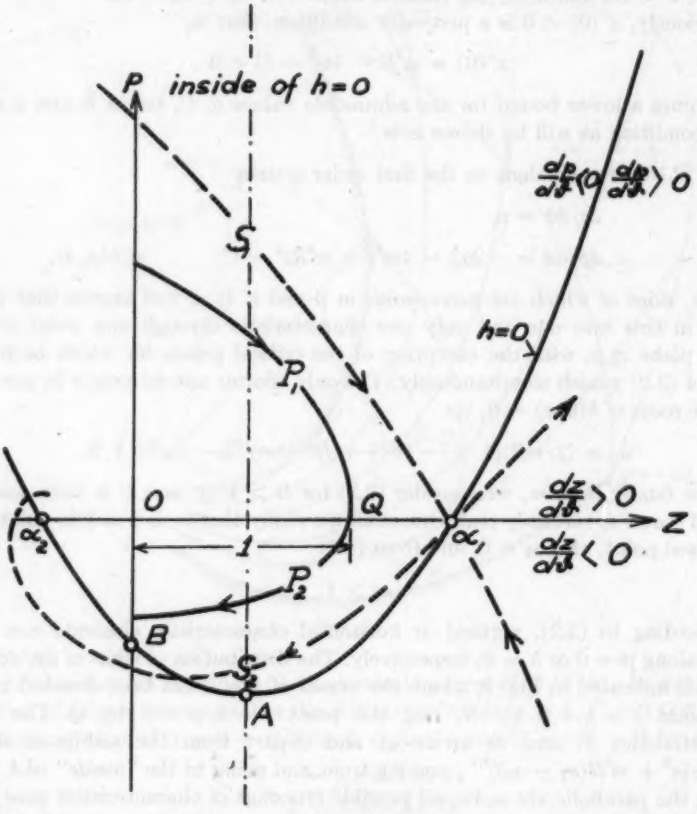

FIGURE 2

fore meets the p-axis. P_i is determined by analytic functions $p_i(\vartheta)$, $z_i(\vartheta)$, and the function $z = z_1(\vartheta)$, $\vartheta \leqq 0$, $z = z_2(\vartheta)$, $\vartheta \geqq 0$ fulfills requirement (a). Note that p_1, z_1 and z_2 are *monotonic* for $z \geqq 0$; in this range z_1 is convex, z_2 starts as a convex arc but will have exactly one inflection point, when P_2 crosses the parabola

[3] Further discussion of S_2 of course show that S_2 spirals into the focus α_2 (Bendixson's second criterion excludes limit cycles in the present case), while S_1 and the remaining two half-characteristics tend to a non-elementary critical point at $y = \pm\infty$.

before meeting the p-axis.—All this hinges on relation (3.4), equivalent to (3.1); *the latter is therefore a sufficient condition for* (a), provided R is large enough.[4]

To study the dependence of Θ on C, we substitute in (2.10) a definite solution $z = z_i(\vartheta; C, R)$ satisfying (a), multiply with $dz_i/d\vartheta$, and integrate from 0 to ϑ. Since $z_i(\vartheta)$ is monotonic, the result may be written as

$$p_i^2 + 4a\int_1^z p_i dz + 4m^2(z^2 - 1) - \tfrac{2}{3}m^3R(z^3 - 1) + 2C(z - 1) = 0 \tag{3.5}$$

and considered as a functional equation for $p_i(z)$ in the independent variable z. Write (3.5) for two different permissible C-values C', C'' and subtract. Then

$$(p_i'^2 - p_i''^2) + 4a\int_1^z (p_i' - p_i'')\,dz = 2(C' - C'')(1 - z), \qquad C' > C'', \text{ say.} \tag{3.6}$$

Now (3.1) implies that, for a sufficiently small δ, $p_i'^2 > p_i''^2$ for $|z - 1| < \delta$. If there were a number $\bar{z} < 1$ such that for the first time $p_i'(z) = p_i''(z)$, then the *first term* of (3.6) would pass from positive values through zero to negative or positive values, when z passes decreasingly through $\bar{z}$. If (3.6) is differentiated with respect to $-z$, then at $z = \bar{z}$ that term gives a non-positive contribution, the second term gives nothing, but the r.h. term gives a positive contribution. Hence there is no such $\bar{z}$, and for the *whole* interval $0 \leqq z < 1$ $p_1' > p_1''$, $p_2' < p_2''$ if $C' > C''$. This implies for

$$\vartheta_i(z) = (-1)^i \int_z^1 dz/|p_i| \tag{3.7}$$

the relation $|\vartheta_i'(z)| < |\vartheta_i''(z)|$. The same argument, applied to a change of R, gives rise to the same inequality, when $R' < R''$; hence we have in particular for $z = 0$

$$\begin{cases} \Theta_1' < \Theta_1'', \quad \Theta_2' < \Theta_2'', \quad \Theta' < \Theta'' \\ \qquad \text{when either } C' > C'' \text{ and } R \text{ unchanged} \\ \qquad\qquad \text{or } R' < R'' \text{ and } C \text{ unchanged.} \end{cases} \tag{3.8}$$

When (3.5) is written for $i = 1, 2$ and subtracted, one obtains

$$p_1^2 - p_2^2 = 4a\int_z^1 (p_1 - p_2)\,dz > 0, \qquad \text{or,} \qquad p_1^2 > p_2^2, \tag{3.9}$$

whence $|\vartheta_2(z)| > |\vartheta_1(z)|$ and in particular

$$\Theta_2 > \Theta_1, \quad \text{and therefore} \quad \Theta_2 > \Theta/2. \tag{3.10}$$

Condition (b) can now be disposed of. Suppose m and a sufficiently large R prescribed, then by (3.1) $m^3R - 4m^2 < C < \infty$. If C approaches its lower

[4] When $m = 1$ and $a = 0$, the same conclusion holds, but the parabola degenerates into a pair of parallel lines, α_2 becomes a vortex point, $z_2(\vartheta) = z_1(-\vartheta)$, and no inflection point can occur for real α_r.

bound, α_1 moves toward 1, P_i in Fig. 2 approaches S_i, and the integral (3.7) becomes divergent at the upper terminus owing to the finite slope of the saddle characteristics. If on the other hand $C \to \infty$, Eq. (4.11) below shows that $\Theta_i \to 0$. Hence Θ as a function of C decreases monotonically from ∞ to 0, and can be made to assume any value between 2π and zero. In the same way, Θ is a monotonically increasing function of R, and $\Theta \to \infty$ when R approaches its upper bound for given C.

4. Estimates for $p_i(z)$; behavior of $p_2(z)$ at large R. Introducing the amount of the curvature of $z(\vartheta)$ at $\vartheta = 0$, viz.

$$c = -z''(0) = C - (m^3R - 4m^2) > 0 \tag{4.1}$$

and the mean value

$$\bar{p}_i = [1/(1-z)]\int_z^1 p_i\,dz, \tag{4.2}$$

one can rewrite Eq. (3.5)

$$p_i^2/(1-z) - 4a\bar{p}_i = -4m^2(1-z) + \tfrac{3}{8}m^3R(1-z)(2+z) + 2c. \tag{4.3}$$

In the interval $I = [0 \leqq z \leqq 1]$

$$\bar{p}_1(z) = k_1(z)p_1\,, \qquad \text{with } 0 \leqq k_1(z) < 1, \qquad z \in I. \tag{4.4}$$

However, p_2 may possess a minimum p^* at $z^* > 0$. In this case, there is again, for $1 \geqq z \geqq z^*$, $\bar{p}_2 = \kappa(z)p_2$ as in (4.4). For $z < z^*$, we can write[5]

$$\bar{p}(z) = [(1-z^*)/(1-z)]\,\kappa(z^*)p^* + [(z^*-z)/(1-z)]\cdot p_m\,, \tag{4.5}$$

where $p^* = p(z^*)$, $p_m = (z^*-z)^{-1}\int_z^{z^*} p\,dz = p[z + \theta(z^*-z)]$, and $0 < \theta \leqq 1$.

Now $z^* \leqq z_A = 2/mR$, hence from a certain R' on

$$0 \leqq z < z + \theta(z^*-z) < z^* \leqq z_A < \epsilon, \qquad R > R'. \tag{4.6}$$

Further,

$$-m^3R(\alpha_1-\alpha_2)^2/8a, = p_A < p^* < p_m < p(z) \leqq p(0) < p_B = -C/2a. \tag{4.7}$$

But $p_A - p_B = -2m/aR$, hence $p(0) - p^* < \epsilon$ for $R > R''$; when we put $p_m = p(z)(1+\epsilon_1)$, $p^* = p(z)(1+\epsilon_2)$, $z = \epsilon_3$, $z^* = \epsilon_4 \geqq \epsilon_3$, where $\epsilon_i < \epsilon$, then (4.5) becomes

$$\begin{aligned}\bar{p}(z) &= p(z)[(1-\epsilon_4)\kappa(z^*)(1+\epsilon_2) + (\epsilon_4-\epsilon_3)(1+\epsilon_1)]/(1-\epsilon_3)\\ &= k_2(z)p(z), \qquad z \in I,\end{aligned} \tag{4.8}$$

where $0 \leqq k_2(z) \leqq 1+\delta$, and $\delta \to 0$, when $R \to \infty$.

We can now show that for sufficiently large R and valid (3.1) none of the curves $p_2(z)$ has a minimum for an inner point of I. Otherwise, Eq. (3.5) with

[5] p stands for p_2 in Eqs. (4.5)–(4.8).

$i = 2$, second term replaced according to (4.8) and $z = 0$, should admit a root $p_2(0)$ smaller than $-C/2a$. This would mean

$$C^2/4a^2 < 2C(1 - k_2(0)) + 8m^3/3, \tag{4.9}$$

which cannot remain valid for $C \to \infty$.

From (4.3) follow the estimation formulas (we write $x = 1 - z$)

$$p_i = 2axK_i(x) - (-1)^i x^{1/2}\{x[\tfrac{4}{3}m^3R(3 - x) + 4(a^2K_i^2(x) - m^2] + 2c\}^{1/2} \tag{4.10}$$

where $K_i(x) = k_i(z)$ are now *both* bounded by 0 and 1, if R is large enough. An alternative form of the brace in (4.10) is

$$\{ \cdots \} = \{4a^2K_i^2(x)(2 - x) + 4m^2x + 2C - \tfrac{4}{3}m^3R(x^2 - 3x + 3)\}. \tag{4.11}$$

5. Transition to the limit $R \to \infty$ at constant Θ. This process may be visualized as a sequence of alternating augmentations of C and R that compensate one another in their action on Θ, cf. (3.8). Thus $C-$ (and $c-$) values for any large R can be determined, when one starts from a given combination of compatible C, R, Θ.

The *general shape of the velocity profile* $z_i(\vartheta)$ for large R at constant Θ can be investigated by means of (4.10). Omitting certain positive terms in the expressions for p_1 and $|p_2|$ given by (4.10), one has

$$\begin{aligned} p_1 &> x[\tfrac{4}{3}m^3R(3 - x) - 4m^3]^{1/2} \\ |p_2| &> x[\tfrac{4}{3}m^3R(3 - x) - 4m^3]^{1/2} - 2a. \end{aligned} \tag{5.1}$$

$xI.$

The square root in (5.1) is real when $R > 2/m$, and for $R > 4/m$ the r.h. sides of (5.1) are monotonically decreasing. Therefore, in the interval $I_\epsilon = \epsilon \leqq x \leqq 1]$, $\epsilon > 0$,

$$[(5.2) \quad |p_i| > 2m\epsilon[mR(3 - \epsilon)/6 - 1]^{1/2} - 2a > 2m\epsilon[mR/3 - 1]^{1/2} - 2,$$

and to any given combination (N, ϵ) an R' can be found so that $|p_i| > N$ for all $x \in I_\epsilon$, when $R > R'$. The velocity profile in the proximity of the channel wall, $\vartheta = -\Theta_1$, stays within the straight lines $\vartheta = -\Theta_1$ and $z = N(\vartheta + \Theta_1)$ at least until $z > 1 - \epsilon$, the analogous statement holding for the wall Θ_2. The ordinate $1 - \epsilon$ is reached at a distance from the wall $\Delta\vartheta < (1 - \epsilon)/N < 1/N$.

Let us construct the sequence of arcs at constant Θ, $z(R_n)$, $R_n \to \infty$, above the *same* segment $\overline{AB}$ of length Θ. Since almost all arcs are convex, our result means that the arcs $z(R_n)$ converge uniformly toward

$$z = 1 \tag{5.3}$$

on a closed subinterval of $\overline{AB}$, not including A or B. Eq. (5.3) is the well-known perfect fluid solution for the problem considered.

Eq. (4.10) shall now be used to obtain the *asymptotic velocity profile* at the wall (boundary layer). For this purpose one should know the behavior of c for

large R. On recalling the definition of c in (4.1), the preceding discussion seems to leave little doubt that $c \to 0$, but a separate proof is necessary, since the maxima of the arcs $z(R_n)$ may in the limit move to A or B.

We shall use (3.7) to connect Θ and c. A non-vanishing lower bound for $|p_i(x)|$ in I being unavailable, it is natural to estimate $p_i(x)x^{-1/2}$. We have, however, to operate with p_2 rather than p_1, because of the possibility $\Theta_1 \to 0$, whereas always $\Theta_2 > \Theta/2$, cf. (3.10). Then by (4.10)

$$(5.4) \qquad |p_2|x^{-1/2} > \{\tfrac{3}{2}m^2Rx(3-x) - 4m^2x + 2c\}^{1/2} - 2ax^{1/2} \equiv f(x;R);$$

$f(x)$ stays positive in I, and a simple discussion shows that of the three possible extremes of $f(x)$ there is always *exactly* one minimum inside I. (That there is *at least* one minimum inside I follows from $f(0) < f(1)$ and $f'(0) = -\infty$.) Let $M(R)$ denote that minimum.

If c is *not* bounded, we can select a sequence R_n such that the associated $c_n \to \infty$ monotonically, and as $f(x;R_{k+1}) > f(x;R_k)$ *also for* $x = 0$, so $M(R_{k+1}) > M(R_k)$; thus $M(R_k) \to \infty$ for all $x \in I$. Now by (3.10), and (3.7) for $z = 0$,

$$(5.5) \qquad 0 < \Theta/2 < \Theta_2 = \int_0^1 \frac{dx}{x^{1/2}\cdot|p_2|x^{-1/2}} < \int_0^1 \frac{dx}{x^{1/2}M(R_n)} = \frac{2}{M(R_n)} \to 0,$$

which is impossible with Θ being a fixed number. Thus c must be *bounded*. This implies by (4.1) and (3.3)

$$(5.6) \qquad \lim C/R = m^2, \qquad \lim \alpha_i = -(-1)^i,$$

and, as a consequence of $\alpha_1 \to 1$, we finally have

$$(5.7) \qquad -c = dp/d\vartheta\,|_{\substack{p=0\\ s=1}} \to dp/d\vartheta\,|_{\substack{p=0\\ s=\alpha_1}} = 0.$$

On using the first relation (5.6), the limiting form of Eq. (2.10) is obtained as

$$(5.8) \qquad z^2 - 1 = 0.$$

In view of (5.3), we may then say that $z(\vartheta)$ converges (with the above specifications) towards one of the solutions of the limiting form of its differential equation.

In the case $m = 1$, $a \neq 0$, it can even be estimated *how fast* c goes to zero. Eq. (2.11) must give $z = 0$ for $\vartheta = \Theta/2$, thus $sn^2(A\Theta/2) = (\xi_1 k^2)^{-1}$. The right hand side of this relation converges toward one, since $k \to 1$ and $\xi_1 \to 1$. The argument of sn approaches therefore K, the complete elliptic integral of first kind belonging to k. From the known expansion of K in terms of k one has $K \sim \log[4/(1-k^2)^{1/2}]$, valid for $k \sim 1$. On replacing here K by $A\Theta/2$ and noticing $A \to (R/2)^{1/2}$, one has $k^2 \sim 1 - 16\exp[-\Theta(R/2)^{1/2}]$. From formulas following (2.11), k^2 may be expanded: $k^2 \sim 1 - 2c/R$; hence the result

$$(5.9) \qquad c = O(R\exp[-\Theta(R/2)^{1/2}]).$$

Since in the present case the coefficients c_{2n} of the expansion of $z(\vartheta)$ about $\vartheta = 0$, $z = 1 - c\vartheta^2/2 + c_4\vartheta^4 + \cdots$, are polynomials of degree n in c and R, divisible by c, one can conclude from (5.9) that c and *all* c_{2n} go to zero when $R \to \infty$.

Using the result $c = O(1)$, one obtains from (4.10)

$$p_i = -(-1)^i(\tfrac{2}{3}m^3)^{1/2}\, x(3-x)^{1/2}R^{1/2} + 2axK_i(x) + O(R^{-1/2}). \tag{5.10}$$

The leading term in (5.10) serves to characterize the behavior of p_i for large R in the practice of fluid dynamics. As is well known, a strict formalization requires the introduction of the stretched y-coordinate

$$\eta = R^{1/2}\Delta\vartheta; \qquad \Delta\vartheta = \Theta_1 + \vartheta \quad \text{or} \quad \Delta\vartheta = \Theta_2 - \vartheta. \tag{5.11}$$

The limiting form of (5.10) is then for either wall

$$dz/d\eta = (\tfrac{2}{3}m^3)^{1/2}(1-z)\,(2+z)^{1/2}, \tag{5.12}$$

and the boundary layer profile is given by

$$\eta = \left(\frac{3}{2m^3}\right)^{1/2} \int_0^z \frac{dz}{(1-z)(2+z)^{1/2}}, \quad \text{or,} \tag{5.13}$$

$$z = 3\tanh^2[\eta(m^3/2)^{1/2} + \operatorname{artanh}(2/3)^{1/2}] - 2.$$

When (5.11) is introduced in (2.10) and transition is made to $R = \infty$ as in (5.8), one obtains the differential equation of the boundary layer

$$(d^2z/d\eta^2) - m^3z^2 + m^3 = 0 \tag{5.14}$$

which is solved by (5.13) for the boundary conditions $z(0) = 0$, $z(\infty) = 1$. This equation is obtained directly from the usual formulas of boundary layer theory when the pressure gradient $d(p/\rho q_0^2)/dx = -m/(1-mx)^3$ is introduced, x being the dimensionless arc length s/r_0 along the wall. The affine transformation $\eta_1 = m^{3/2}\eta$ makes all boundary layer profiles congruent.

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Using the result $z = O(1)$, one obtains from (7.10)

(5.10) [illegible]

The leading term in (5.10) serves to characterize the behavior of [illegible] in the presence of [illegible] as well as [illegible] requires the introduction of the stretched coordinates

(5.11) [illegible]

The [illegible] term of (5.10) is then [illegible]

(5.12) [illegible]

and the boundary layer profile is given by

(5.13) [illegible]

When (5.11) is introduced in (5.10) and integration is made [illegible], one obtains the differential equation of the boundary layer

(5.14) [illegible]

which is solved by (5.13) for the boundary conditions [illegible]. This solution is obtained directly from the [illegible] of boundary layer theory when the pressure gradient [illegible] being the dimensionless arc length [illegible] along the wall. The other transformation [illegible] makes all boundary layer profiles congruent.

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THE BEHAVIOR AT INFINITY OF THE POTENTIAL FUNCTION OF A TWO DIMENSIONAL SUBSONIC COMPRESSIBLE FLOW*

By G. S. S. Ludford

1. Introduction. 1.1 *Type of flow considered:* We shall discuss the disturbance of a uniform compressible stream by a system of bodies which may be subject to surface suction. The resulting flow is assumed inviscid, irrotational, steady and free from turbulence; in addition we limit the analysis to cases in which the Helmholtz type of flow is absent; we assume the incident stream is sufficiently subsonic so that shocks do not occur.

If we denote by (r, α) the polar coordinates in the physical plane, then it is known that for an incompressible fluid under the above conditions the potential function may be expanded in the form

$$\varphi = q_0 r \cos(\alpha - \theta_0) + (m/2\pi) \log r + (\Gamma/2\pi)\alpha + \sum_{n=1}^{\infty} [a_n(\alpha)/r^n] \tag{1.1.1}$$

where q_0, θ_0 are the magnitude and direction, respectively, of the velocity vector at infinity, m is the source strength and Γ the circulation in the stream (ref. 1)—we shall use suffix zero to denote values at infinity. We wish to show that in the case of a compressible fluid the expansion for φ is:

$$\begin{aligned}\varphi = {} & q_0 r \cos(\alpha - \theta_0) + (m/2\pi\beta) \log r + (\Gamma/2\pi) \tan^{-1}[\beta \tan(\alpha - \theta_0)] \\ & + \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} a_{np}(\alpha) (\log r/r)^n (1/r^p)\end{aligned} \tag{1.1.2}$$

with $a_{np}(\alpha)$ periodic, and $\beta^2 = 1 - M_0^2$, provided not both m, Γ are zero. If $m = \Gamma = 0$, then

$$\varphi = q_0 r \cos(\alpha - \theta_0) + \sum_{n=1}^{\infty} [a_n(\alpha)]/r^n, \qquad a_n(\alpha) \text{ periodic.} \tag{1.1.3}$$

These results can be obtained formally by a discussion of the non-linear differential equation satisfied by φ in the physical plane. We give here a rigorous derivation, making use of the linear differential equation satisfied by φ in the pseudo-logarithmic plane.

Historically, an approximation to the first few terms of this expansion was obtained for a particular case by H. Lamb (ref. 5). That this type of infinite expansion is needed was first conjectured by H. Bateman (ref. 6) though he included terms in $(\log r)^m/r^n$ with $m > n$, which do not appear in (1.1.2) and (1.1.3).

Having established these last two expansions, we may then proceed to determine the lift and moment as given by (4.2.1) and (4.2.2). The former has been given previously by many authors, see for instance (ref. 4), but in each case without justification of the expansion for φ used.

1.2 *The Logarithmic and Pseudo-Logarithmic Planes:* In a paper by Bergman (ref. 2), which forms the basis of our discussion, it is shown that in the logarithmic plane (1.1.1) takes the form:

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$$\varphi = (mi/4\pi)\,(\zeta^{-1} - \bar{\zeta}^{-1}) + (\Gamma/4\pi)\,(\zeta^{-1} + \bar{\zeta}^{-1})$$
$$-(m/4\pi)\,(\log\zeta + \log\bar{\zeta}) + (i\Gamma/4\pi)\log(\zeta/\bar{\zeta}) \tag{1.2.1}$$
$$+ \sum_{n=0}^{\infty} d_n\zeta^n + \sum_{n=0}^{\infty}\bar{d}_n\bar{\zeta}^n$$

where

$$\begin{cases} \zeta = Z - Z_0, \quad Z = \theta + i\lambda \\ \lambda = \log q, \text{ and the } d_n \text{ are constants, provided not both of} \\ \quad m,\ \Gamma \text{ are zero.} \end{cases} \tag{1.2.1a}$$

If $m = \Gamma = 0$, but $a_0(\alpha) \not\equiv 0$, it can be shown similarly that

$$\varphi = a_0\zeta^{-1/2} + \bar{a}_0\bar{\zeta}^{-1/2} + \sum_{n=0}^{\infty} e_n\zeta^{n/2} + \sum_{n=0}^{\infty}\bar{e}_n\bar{\zeta}^{n/2} \tag{1.2.2}$$

where a_0 depends on the constants in $a_0(\alpha)$, and is non-zero. We note immediately the difference between the two cases. For m, Γ not both zero (hereafter called the irregular case, since the complex potential does not have a Laurent series expansion) φ has a first order singularity, whereas for $m = \Gamma = 0$ (the regular case) it has a half order singularity. Moreover we note that φ and its conjugate ψ satisfy:

$$\varphi_{z\bar{z}} = 0, \tag{1.2.3}$$

$$\psi_{z\bar{z}} = 0 \tag{1.2.4}$$

with

$$\varphi = i\int(\psi_z\,dZ - \psi_{\bar{z}}\,d\bar{Z}) \tag{1.2.5}$$

In the case of a compressible fluid, it can be shown that in the so-called pseudo-logarithmic plane (θ, λ) where now

$$\lambda = \tfrac{1}{2}\log\left[\frac{1 - (1 - M^2)^{1/2}}{1 + (1 - M^2)^{1/2}}\cdot\left(\frac{1 + h(1 - M^2)^{1/2}}{1 - h(1 - M^2)^{1/2}}\right)^{1/h}\right] \tag{1.2.6}$$

with

$$h = [(\gamma - 1)/(\gamma + 1)]^{1/2}; \qquad \gamma > 1$$

the potential function φ and its conjugate ψ satisfy

$$\varphi^*_{z\bar{z}} + P\varphi^* = 0 \qquad \text{with } \varphi^* = H\varphi \tag{1.2.7}$$

$$\psi^*_{z\bar{z}} + F\psi^* = 0 \qquad \text{with } \psi^* = H^{-1}\psi \tag{1.2.8}$$

with

$$\varphi = i\int(l^{1/2}\psi_z\,dZ - l^{1/2}\psi_{\bar{z}}\,d\bar{Z}) \tag{1.2.9}$$

The functions P, F, H, l are listed in Bergman's paper (as also is all other notation not explicitly given in this paper), and depend on $\lambda = (Z - \bar{Z})/2i$ only.

The whole discussion now hinges upon the fact that the psuedo-logarithmic

plane degenerates into the logarithmic plane as the compressibility tends to zero, i.e. the λ of (1.2.6) tends to the λ of (1.2.1a). Thus since the leading terms in (1.2.1) and (1.2.2) are certain definite singular solutions of (1.2.3) we may define the compressible potential function to have leading terms which are the corresponding (definite) solutions of (1.2.7). Essentially a complete correspondence of singularities in the two planes is set up.

For further details of the completeness of the correspondence, the reader is referred to the bibliography of Bergman's work given in ref. 2.

2. Irregular Case. 2.1 *The Singularities:* The following formulae, not given by Bergman, will be constantly used and may be listed for reference:

$$\text{(2.1.1)} \qquad \begin{cases} l_0 &= \beta/\rho_0 \neq 0, \infty \quad \text{for} \quad M_0 < 1 \\ l_1 &= (M_0^2/\rho_0\beta^2 q_0^2)(\beta^2 q_0^2 - M_0^2) \\ l_{-1} &= -(\rho_0 M_0^2/\beta^4 q_0^2)\ (\beta^2 q_0^2 - M_0^2) \end{cases}$$

$$\text{(2.1.2)} \qquad \begin{cases} h_0 &= \rho_0^{1/2}/\beta^{1/2} \neq 0, \infty \quad \text{for} \quad M_0 < 1. \\ h_1 &= -\tfrac{1}{2}(M_0^2 \rho_0^{1/2}/\beta^{7/2} q_0^2)(\beta^2 q_0^2 - M_0^2) \\ H &= l^{-1/4} \end{cases}$$

Moreover we shall use $\Sigma(\zeta_1, \cdots \zeta_k)$ to denote a k-infinite power series in the k independent variables $(\zeta_1, \cdots \zeta_k)$, known to be uniformly and absolutely convergent in some neighborhood of the origin. By Σ_m we denote a Σ which definitely does not contain terms up to and including the m^{th} order, and by Σ^m a Σ_{m-1} which definitely does contain a term of the m^{th} order. Thus the symbol Σ will be used to denote several functions of quite definite form, but whose exact form is not important. Finally ζ and $\bar{\zeta}$ etc. will be in general considered as independent complex variables though in the interpretation of the results we shall need them to be conjugate, and will note when they are.

Bergman now considers the following singularities:

(a) The fundamental solution of (1.2.8)

$$\text{(2.1.3)} \qquad \begin{aligned} \psi^{(L,1)} &= \tfrac{1}{2}A(\log\zeta + \log\bar{\zeta}) + B \\ &= \tfrac{1}{2}(\log\zeta + \log\bar{\zeta})[h_0 + \Sigma_0(\zeta, \bar{\zeta})] + \Sigma(\zeta, \bar{\zeta}) \end{aligned}$$

so that from (1.2.5) with

$$\text{(2.1.4)} \qquad l^{1/2} = l_0 + \Sigma_0(\zeta, \bar{\zeta})$$

we have

$$\text{(2.1.5)} \qquad \varphi^{(L,1)} = \tfrac{1}{2}ih_0 l_0 \log(\zeta/\bar{\zeta}) + (\log\zeta + \log\bar{\zeta})\Sigma_0(\zeta, \bar{\zeta}) + \Sigma(\zeta, \bar{\zeta})$$

where we used the fact that the integrand of (1.2.5) is single-valued when $\zeta, \bar{\zeta}$ are conjugate. We note that $\varphi^{(L,1)}$ reduces to

$$\text{(2.1.6)} \qquad \varphi_0^{(L,1)} = \tfrac{1}{2}i \log(\zeta/\bar{\zeta})$$

as the compressibility tends to zero. We omit a factor $\rho_0^{-1/2}$.

(b) The fundamental solution of (1.2.7)

$$\begin{aligned}\varphi^{(L,2)} &= \tfrac{1}{2}C\,(\log\zeta + \log\bar{\zeta}) + D \\ &= \tfrac{1}{2}\,(\log\zeta + \log\bar{\zeta})[h_0^{-1} + \Sigma_0(\zeta,\bar{\zeta})] + \Sigma(\zeta,\bar{\zeta})\end{aligned} \tag{2.1.7}$$

This singularity reduces to

$$\varphi_c^{(L,2)} = \tfrac{1}{2}\,(\log\zeta + \log\bar{\zeta}) \tag{2.1.8}$$

as the compressibility tends to zero.

(c) The first order singularity:

$$\begin{aligned}\varphi^{(1,1)} &= (\partial/\partial\zeta)\,\varphi^{(L,1)} + (\partial/\partial\bar{\zeta})\,\varphi^{(L,1)} \\ &= il^{1/2}[\tfrac{1}{2}A(\zeta^{-1} - \bar{\zeta}^{-1}) + \tfrac{1}{2}(A_\zeta - A_{\bar{\zeta}})\log\zeta\bar{\zeta} + (B_\zeta - B_{\bar{\zeta}})] \\ &= (1/\zeta)[\tfrac{1}{2}ih_0l_0 + \Sigma_0(\bar{\zeta})] - (1/\bar{\zeta})[\tfrac{1}{2}ih_0l_0 + \Sigma_0(\zeta)] \\ &\quad + (\log\zeta + \log\bar{\zeta})[\tfrac{1}{2}h_1l_0 + \Sigma_0(\zeta,\bar{\zeta})] + \Sigma(\zeta,\bar{\zeta})\end{aligned} \tag{2.1.9}$$

which reduces to

$$\varphi_c^{(1,1)} = \tfrac{1}{2}i(\zeta^{-1} - \bar{\zeta}^{-1}) \tag{2.1.10}$$

(d) The first order singularity:

$$\begin{aligned}\varphi^{(1,2)} &= (\partial/\partial\zeta)\varphi^{(L,2)} + (\partial/\partial\bar{\zeta})\varphi^{(L,2)} \\ &= \tfrac{1}{2}C(\zeta^{-1} + \bar{\zeta}^{-1}) + \tfrac{1}{2}(C_\zeta + C_{\bar{\zeta}})\log\zeta\bar{\zeta} + D_\zeta + D_{\bar{\zeta}} \\ &= \zeta^{-1}[(1/2h_0) + \Sigma_0(\bar{\zeta})] + \bar{\zeta}^{-1}[(\tfrac{1}{2}h_0) + \Sigma_0(\zeta)] \\ &\quad + (\log\zeta + \log\bar{\zeta})\Sigma_0(\zeta,\bar{\zeta}) + \Sigma(\zeta,\bar{\zeta})\end{aligned} \tag{2.1.11}$$

which reduces to

$$\varphi_c^{(1,2)} = \tfrac{1}{2}(\zeta^{-1} + \bar{\zeta}^{-1}) \tag{2.1.12}$$

We see from (2.1.6), (2.1.8), (2.1.10) and (2.1.12) that the leading terms in (1.2.1) are:

$$(m/2\pi)\varphi_c^{(1,1)} + (\Gamma/2\pi)\varphi_c^{(1,2)} + (\Gamma/2\pi)\varphi_c^{(L,1)} - (m/2\pi)\varphi_c^{(L,2)} \tag{2.1.13}$$

Thus Bergman proposes to use a form

$$\varphi = \mu^{(1,1)}\varphi^{(1,1)} + \mu^{(1,2)}\varphi^{(1,2)} + \mu^{(L,1)}\varphi^{(L,1)} + \mu^{(L,2)}\varphi^{(L,2)} + \varphi' \tag{2.1.14}$$

for the compressible potential, with the μ's constant and φ' a regular function. Just as (2.1.13) falls into two pairs of terms determined by m and Γ respectively, so in (2.1.14), as will be shown. The correspondence between the potential functions in the logarithmic and pseudo-logarithmic planes will thus be completed.

With (2.1.5), (2.1.7), (2.1.9) and (2.1.11) we obtain from (2.1.14):

$$\begin{aligned}\varphi = \zeta^{-1}\,[a + \Sigma_0(\bar{\zeta})] + \bar{\zeta}^{-1}\,[\bar{a} + \Sigma_0(\zeta)] \\ + (\log\zeta + \log\bar{\zeta})[b + \Sigma_0(\zeta,\bar{\zeta})] + c\log(\zeta/\bar{\zeta}) + \Sigma(\zeta,\bar{\zeta})\end{aligned} \tag{2.1.15}$$

with

$$(2.1.16)\qquad \begin{cases} a = (1/2h_0)[\mu^{(1,2)} + i\mu^{(1,1)}] \neq 0 \\ b = \tfrac{1}{2}h_1 l_0\ \mu^{(1,1)} + (1/2h_0)\mu^{(L,2)} \\ c = (i/2h_0)\ \mu^{(L,1)} \end{cases}$$

2.2 *Transformation to the Physical Plane:* A laborious though straightforward calculation gives Bergman's T-periods explicitly as:

$$(2.2.1)\qquad \begin{cases} X^{(1,1)} = -T\beta \sin\theta_0 & Y^{(1,1)} = T\beta \cos\theta_0 \\ X^{(1,2)} = T \cos\theta_0 & Y^{(1,2)} = T \sin\theta_0 \\ X^{(L,1)} = -T\beta \cos\theta_0 & Y^{(L,1)} = -T\beta \sin\theta_0 \\ X^{(L,2)} = -T \sin\theta_0 & Y^{(L,2)} = T \cos\theta_0 \end{cases}$$

with

$$(2.2.1a)\qquad T = 2\pi/q_0\,\rho_0^{1/2}\beta^{1/2}$$

Hence it may be shown that the "consistency relations"

$$(2.2.2)\qquad \begin{cases} \mu^{(1,1)}X^{(1,1)} + \mu^{(1,2)}X^{(1,2)} + \mu^{(L,1)}X^{(L,1)} + \mu^{(L,2)}X^{(L,2)} = 0 \\ \mu^{(1,1)}Y^{(1,1)} + \mu^{(1,2)}Y^{(1,2)} + \mu^{(L,1)}Y^{(L,1)} + \mu^{(L,2)}Y^{(L,2)} = 0 \end{cases}$$

imply

$$(2.2.3)\qquad \mu^{(1,2)}/\mu^{(L,1)} = -\mu^{(L,2)}/\mu^{(1,1)} = \beta$$

which justifies the statement made about (2.1.14). We see that (2.1.13) satisfies (2.2.3) with $\beta = 1$.

The constants in (2.1.16) may be written, in view of (2.2.3) in the form

$$(2.2.4)\qquad \begin{cases} a = a_1 + ia_2 \neq 0 & a_1 = \beta^{1/2}\mu^{(1,2)}/2\rho_0^{1/2} \qquad a_2 = \beta^{1/2}\mu^{(1,1)}/2\rho_0^{1/2} \\ b = (a_2/h_0)(h_1 - \beta h_0), & c = ia_1/\beta \end{cases}$$

Now substituting (2.1.15) with (2.2.4) into the transformation equations listed by Bergman on pp. 466–7 of his paper, another tedious calculation (for which the necessary expansions are used in the evaluation of the T-periods) yields the functions $x(\zeta, \bar{\zeta})$, $y(\zeta, \bar{\zeta})$ in the form:

$$(2.2.5)\qquad \begin{aligned} x = {} & (ad/\zeta)[1 + \Sigma_0(\bar{\zeta})] + (\overline{ad}/\bar{\zeta})\,[1 + \Sigma_0(\zeta)] \\ & + (\log\zeta + \log\bar{\zeta})[g + \Sigma_0(\zeta, \bar{\zeta})] + \Sigma(\zeta, \bar{\zeta}) \end{aligned}$$

$$(2.2.6)\qquad \begin{aligned} y = {} & (ad^*/\zeta)\,[1 + \Sigma_0(\bar{\zeta})] + (\overline{ad^*}/\bar{\zeta})\,[1 + \Sigma_0(\zeta)] \\ & + (\log\zeta + \log\bar{\zeta})[h + \Sigma_0(\zeta, \bar{\zeta})] + \Sigma(\zeta, \bar{\zeta}) \end{aligned}$$

with

$$(2.2.7)\qquad \begin{cases} d &= (1/\beta q_0)(\beta \cos\theta_0 + i \sin\theta_0) \\ d^* &= (1/\beta q_0)(\beta \sin\theta_0 - i \cos\theta_0) \\ g &= (M_0^2/2\beta^4 q_0^3)(M_0^2 + \beta^2 q_0^2)(\beta a_2 \cos\theta_0 - a_1 \sin\theta_0) \\ h &= (M_0^2/2\beta^4 q_0^2)(M_0^2 + \beta^2 q_0^2)(a_1 \cos\theta_0 + \beta a_2 \sin\theta_0) \end{cases}$$

Next we define two new variables z, $\bar{z}$ (which are the analogues of the complex variable and its conjugate of incompressible flow) by the formula

$$(2.2.8)\qquad z = (q_0\beta/i)(\overline{d^*}x - \bar{d}y) = r[\cos(\alpha - \theta_0) + i\beta \sin(\alpha - \theta_0)]$$

We see that for ζ, $\bar{\zeta}$ conjugate, i.e. r, α real, (2.2.8) is just the transformation used in the linearised theory of compressible flow. From (2.2.5), (2.2.6) and (2.2.8) we have:

$$(2.2.9)\qquad \begin{aligned} z = (2a/q_0\zeta)[1 + \Sigma_0(\bar{\zeta})] + (\zeta/\bar{\zeta})\Sigma(\zeta) \\ + k\bar{a}\,(\log\zeta + \log\bar{\zeta})[1 + \Sigma_0(\zeta, \bar{\zeta})] + \Sigma(\zeta, \bar{\zeta}) \end{aligned}$$

and similarly for $\bar{z}$. We note that by this linear transformation we have dissociated the leading terms $1/\zeta$, $1/\bar{\zeta}$ in (2.2.5) and (2.2.6).

2.3 *The Potential Function at Infinity:* We now have to invert (2.2.9) to obtain ζ, $\bar{\zeta}$ as functions of z, $\bar{z}$. Then substitution in (2.1.15) will give the required expansion. To do this we replace $\zeta/\bar{\zeta}$ and $(\log\zeta + \log\bar{\zeta})$ in (2.2.9) by η/ζ and ξ/ζ, and consider ξ, η, ζ as independent variables. Similarly we define $\bar{\xi}$, $\bar{\eta}$. We then have generalized definitions of z, $\bar{z}$ and denote by w, $\bar{w}$ the reciprocals of these. Thus

$$(2.3.1)\qquad w = (q_0/2a)\zeta[1 + \Sigma^1(\xi, \eta, \zeta, \bar{\zeta})]$$

From the generalized z, $\bar{z}$ we may form:

$$(2.3.2)\qquad \begin{aligned} &\frac{\log z + \log\bar{z} - \log(4a\bar{a}/q_0^2)}{z} \\ &= \frac{-q_0\zeta}{2a}(\log\zeta + \log\bar{\zeta})[1 + \Sigma^1(\xi, \eta, \zeta, \bar{\zeta})] + \Sigma^2(\xi, \bar{\xi}, \eta, \bar{\eta}, \zeta, \bar{\zeta}) \end{aligned}$$

$$(2.3.3)\qquad \bar{z}/z^2 = (q_0\bar{a}/2a^2)(\zeta^2/\bar{\zeta})[1 + \Sigma^1(\xi, \bar{\xi}, \eta, \bar{\eta}, \zeta, \bar{\zeta})]$$

and replacing $\zeta(\log\zeta + \log\bar{\zeta})$ by ξ in (2.3.2) and $\zeta^2/\bar{\zeta}$ by η in (2.3.3) we define two new dependent functions u, v respectively:

$$(2.3.4)\qquad u = -(q_0\xi/2a) + \Sigma^2(\xi, \bar{\xi}, \eta, \bar{\eta}, \zeta, \bar{\zeta})$$

$$(2.3.5)\qquad v = (q_0\bar{a}/2a^2)\eta[1 + \Sigma^1(\xi, \bar{\xi}, \eta, \bar{\eta}, \zeta, \bar{\zeta})]$$

and similarly $\bar{u}$, $\bar{v}$. We thus have constructed six functions u, $\bar{u}$, v, $\bar{v}$, w, $\bar{w}$ of six independent variables ξ, $\bar{\xi}$, η, $\bar{\eta}$, ζ, $\bar{\zeta}$. Moreover from (2.3.1), (2.3.4) and (2.3.5) we have

$$(2.3.6)\qquad \frac{\partial(u,\bar{u},v,\bar{v},w,\bar{w})}{\partial(\xi,\bar{\xi},\eta,\bar{\eta},\zeta,\bar{\zeta})} = \frac{q_0^6}{2^6a^3\bar{a}^3} \neq 0 \quad \text{at} \quad \xi = \bar{\xi} = \eta = \bar{\eta} = \zeta = \bar{\zeta} = 0$$

Hence we may invert (2.3.1), (2.3.4) and (2.3.5) to give

$$(2.3.7)\qquad \begin{cases} \xi = -(2a/q_0)u + \Sigma^2(u,\bar{u},v,\bar{v},w,\bar{w}) \\ \eta = (2a^2/q_0\bar{a})v[1 + \Sigma^1(u,\bar{u},v,\bar{v},w,\bar{w})] \\ \zeta = (2a/q_0)w[1 + \Sigma^1(u,\bar{u},v,\bar{v},w,\bar{w})] \end{cases}$$

and similarly $\bar{\xi}$, η, $\bar{\zeta}$. A property of this inversion is that if we form from (2.3.7)

$$(2.3.8)\qquad 1/\zeta = (q_0/2a)(1/w)[1 + \Sigma^1(u,\bar{u},v,\bar{v},w,\bar{w})]$$

then the series Σ^1 occurring in the bracket contains no terms in $u^m\bar{u}^n$ except for $m = 1, n = 0$.

The formulae (2.3.7) are valid for the particular values $\zeta(\log\zeta + \log\bar{\zeta})$ and $\zeta^2/\bar{\zeta}$ of ξ, η respectively. Thus for ζ, $\bar{\zeta}$ small enough we have inverted (2.2.9) to give

$$(2.3.9)\qquad \zeta = (2a/q_0z)[1 + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}k_{mn}(\alpha)\,(\log r/r)^m(1/r)^n]$$

where we have used (2.2.8). We note that $k_{mn}(\alpha)$ and derivatives are continuous periodic functions of α. Substituting (2.3.9) into (2.1.15) and remembering (2.2.4) we have for ζ, $\bar{\zeta}$ conjugates:

$$(2.3.10)\qquad \begin{aligned} \varphi &= q_0r\cos(\alpha - \theta_0) + (m/2\pi\beta)\log r \\ &\quad + (\Gamma/2\pi)\tan^{-1}[\beta\tan(\alpha - \theta_0)] + \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}a_{np}(\alpha)(\log r/r)^n(1/r)^p \end{aligned}$$

with

$$(2.3.11)\qquad m/2\pi\beta = \mu^{(1,1)}/\beta^{1/2}\rho_0^{1/2}; \qquad \Gamma/2\pi = \mu^{(1,2)}/\beta^{1/2}\rho_0^{1/2}$$

where we have used the property stated about (2.3.8). We can infer that the $a_{np}(\alpha)$ are continuous and periodic functions of α, together with their derivatives, and that (2.3.10) is uniformly and absolutely convergent for r large enough.

Further we may easily deduce that the series (2.3.10) can be differentiated term by term at will, and that the resulting series have similar convergence properties. This fact is understood in §4.

3. Regular Case. 3.1 *The Singularities:* The potential (1.2.2) is defined on a Riemann surface of two sheets with branch point at $Z = Z_0$ in the logarithmic plane. If we write

$$(3.1.1)\qquad \sigma = \zeta^{1/2} = (Z - Z_0)^{1/2}$$

then

$$(3.1.2)\qquad \varphi = (a_0/\sigma) + (\bar{a}_0/\bar{\sigma}) + \sum_{n=0}^{\infty}e_n\sigma^n + \sum_{n=0}^{\infty}\bar{e}_n\bar{\sigma}^n$$

is single-valued in the σ-plane. Using the same transformation (3.1.1) of the pseudo-logarithmic plane, we look for a compressible potential which is the

analogue of (3.1.2). Thus we must discuss fundamental solutions and derivative solutions in the σ-plane in a similar manner to that of Bergman in the ζ-plane for the irregular case. In the former plane (1.2.7), (1.2.8) and (1.2.9) become

$$\varphi^*_{\sigma\bar\sigma} + R\varphi^* = 0 \quad \text{with} \quad \varphi^* = H\varphi \tag{3.1.3}$$

$$\psi^*_{\sigma\bar\sigma} + G\psi^* = 0 \quad \text{with} \quad \psi^* = H^{-1}\psi \tag{3.1.4}$$

with

$$\varphi = i\int (l^{1/2}\psi_\sigma\, d\sigma - l^{1/2}\psi_{\bar\sigma}\, d\bar\sigma) \tag{3.1.5}$$

$$\begin{cases} R = 4\sigma\bar\sigma P[(\sigma^2 - \bar\sigma^2 + Z_0 - Z_0)/2i] \\ G = 4\sigma\bar\sigma F[(\sigma^2 - \bar\sigma^2 + Z_0 - \overline{Z}_0)/2i] \end{cases} \tag{3.1.6}$$

and H, l are now functions of $[(\sigma^2 - \bar\sigma^2 + Z_0 - \overline{Z}_0)/2i]$ only.

We may list the required singularities as follows:

(a) The fundamental solution of (3.1.4) at $(\sigma_0, \bar\sigma_0)$:

$$\psi_+^{(L,1)} = \tfrac{1}{2}A_+ [\log(\sigma - \sigma_0) + \log(\bar\sigma - \bar\sigma_0)] + B_+$$

where A_+, B_+ are given by the same formulae as A, B of Bergman's paper except that G is substituted for F and σ, $\bar\sigma$ for Z, $\overline{Z}$. For $\sigma_0 = \bar\sigma_0 = 0$, $A_+ = A$ and $B_+ = \tfrac{1}{2}B$ and hence

$$\psi_+^{(L,1)} = \psi_M = \tfrac{1}{2}\psi^{(L,1)} \tag{3.1.7}$$

From (3.1.7) and (3.1.5) we obtain in this case

$$\varphi_+^{(L,1)} = \varphi_M = \tfrac{1}{2}\varphi^{(L,1)} \tag{3.1.8}$$

(b) The fundamental solution of (3.1.3) at $(\sigma_0, \bar\sigma_0)$:

$$\varphi_+^{(L,2)} = \tfrac{1}{2}C_+ [\log(\sigma - \sigma_0) + \log(\bar\sigma - \bar\sigma_0)] + D_+ \tag{3.1.9}$$

where C_+ and D_+ are given by the formulae analogous to the C and D of Bergman's paper, namely:

$$\begin{aligned} HC_+ &= 1 - \int_{\sigma_0}^{\sigma}\int_{\bar\sigma_0}^{\bar\sigma} R\, d\sigma_1\, d\bar\sigma_1 \\ &\quad + \int_{\sigma_0}^{\sigma}\int_{\bar\sigma_0}^{\bar\sigma} R\left[\int_{\sigma_0}^{\sigma_1}\int_{\bar\sigma_0}^{\bar\sigma_1} R\, d\sigma_2\, d\bar\sigma_2\right] d\sigma_1\, d\bar\sigma_1 + \cdots \\ HD_+ &= \int_{\sigma_0}^{\sigma}\int_{\bar\sigma_0}^{\bar\sigma} L\, d\sigma_1\, d\bar\sigma_1 - \int_{\sigma_0}^{\sigma}\int_{\bar\sigma_0}^{\bar\sigma} R\left[\int_{\sigma_0}^{\sigma_1}\int_{\bar\sigma_0}^{\bar\sigma_1} L\, d\sigma_2\, d\bar\sigma_2\right] d\sigma_1\, d\bar\sigma_1 + \cdots \end{aligned} \tag{3.1.10}$$

with

$$L = -\frac{1}{\bar\sigma - \bar\sigma_0}\frac{\partial(HC_+)}{\partial\sigma} - \frac{1}{\sigma - \sigma_0}\frac{\partial(HC_+)}{\partial\bar\sigma} \tag{3.1.10a}$$

Clearly C_+ and D_+ are analytic functions of $(\sigma, \bar\sigma)$ in some neighborhood of

$(\sigma_0, \bar{\sigma}_0)$. Similarly to (a) we have for $\sigma_0 = \bar{\sigma}_0 = 0$

$$\varphi_+^{(L,2)} = \varphi_L = \tfrac{1}{2}\varphi^{(L,2)} \tag{3.1.11}$$

(c) The first order singularity at $(\sigma_0, \bar{\sigma}_0)$:

$$\begin{aligned}\varphi_+^{(1,1)} &= (\partial/\partial\sigma_0)\varphi_+^{(L,1)} + (\partial/\partial\bar{\sigma}_0)\varphi_+^{(L,1)} \\ &= -\tfrac{1}{2}C_+\left[\frac{1}{\sigma - \sigma_0} + \frac{1}{\bar{\sigma} - \bar{\sigma}_0}\right] + \tfrac{1}{2}(\partial C_+/\partial\sigma_0 + \partial C_+/\partial\bar{\sigma}_0) \\ &\quad \times [\log(\sigma - \sigma_0) + \log(\bar{\sigma} - \bar{\sigma}_0)] + (\partial D_+/\partial\sigma_0 + \partial D_+/\partial\bar{\sigma}_0)\end{aligned}$$

Now from (3.1.10) it is easily proved that for $\sigma_0 = \bar{\sigma}_0 = 0$

$$\partial C_+/\partial\sigma_0 = \partial C_+/\partial\bar{\sigma}_0 = 0 \tag{3.1.12}$$

and hence in this case, from (3.1.11) and (3.1.10):

$$\begin{aligned}\varphi_+^{(1,1)} = \varphi_1 = (1/\sigma)[-(1/2h_0) + \Sigma_0(\bar{\sigma}^2)] \\ + (1/\bar{\sigma})[-(1/2h_0) + \Sigma_0(\sigma^2)] + \Sigma(\sigma, \bar{\sigma})\end{aligned} \tag{3.1.13}$$

(d) The first order singularity at $(\sigma_0, \bar{\sigma}_0)$:

$$\varphi_+^{(1,2)} = i[(\partial/\partial\sigma_0)\varphi_+^{(L,1)} - (\partial/\partial\bar{\sigma}_0)\varphi_+^{(L,1)}] \tag{3.1.14}$$

which reduces as in (c) to:

$$\begin{aligned}\varphi_+^{(1,2)} = \varphi_2 = (1/\sigma)[-(i/2h_0) + \Sigma_0(\bar{\sigma}^2)] \\ + (1/\bar{\sigma})[(i/2h_0) + \Sigma_0(\sigma^2)] + \Sigma(\sigma, \bar{\sigma})\end{aligned} \tag{3.1.15}$$

for $\sigma_0 = \bar{\sigma}_0 = 0$.

The first order singularities have been obtained in a slightly different way from that in §2. Since however there are only two independent singularities we may choose two such the most easily obtained, ensuring that they are real.

As the compressibility tends to zero, the above singularities tend to

$$\begin{cases}\varphi_M^0 = \tfrac{1}{2}i\log(\sigma/\bar{\sigma}) \\ \varphi_L^0 = \tfrac{1}{2}i(\log\sigma + \log\bar{\sigma}) \\ \varphi_1^0 = -\tfrac{1}{2}(\sigma^{-1} + \bar{\sigma}^{-1}) \\ \varphi_2^0 = -\tfrac{1}{2}i(\sigma^{-1} - \bar{\sigma}^{-1})\end{cases} \tag{3.1.16}$$

where we use (3.1.8) and (3.1.11) for the first two and (3.1.10) together with the fact that $R \to 0$ as the compressibility tends to zero for the last two. Again a factor $\rho_0^{1/2}$ is omitted. We see from (3.1.16) that (3.1.12) may be written

$$\varphi = \mu_1^0\varphi_1^0 + \mu_2^0\varphi_2^0 + \mu_L^0\varphi_L^0 + \mu_M^0\varphi_M^0 + \varphi' \tag{3.1.17}$$

where

$$\mu_1^0 = -2\mathrm{Re}\,(a_0), \quad \mu_2^0 = -2\,\mathrm{Im}\,(a_0), \quad \mu_L^0 = \mu_M^0 = 0 \tag{3.1.18}$$

and φ' is a regular function of $(\sigma, \bar{\sigma})$. In analogy to (3.1.17) we shall write

$$\varphi = \mu_1\varphi_1 + \mu_2\varphi_2 + \mu_L\varphi_L + \mu_M\varphi_M + \varphi' \tag{3.1.19}$$

with the μ's constant. Just as in (3.1.17) $\mu_L^0 = \mu_M^0 = 0$, and logarithmic terms are missing, so in (3.1.19) we shall have $\mu_L = \mu_M = 0$, and correspondence will be complete.

Assuming the latter fact, we may write from (3.1.13), (3.1.15) and (3.1.19):

$$\varphi = (a/\sigma)[1 + \Sigma_0(\bar{\sigma}^2)] + (\bar{a}/\bar{\sigma})[1 + \Sigma_0(\sigma^2)] + \Sigma(\sigma, \bar{\sigma}) \tag{3.1.20}$$

with

$$a = -(1/2h_0)(\mu_1 + i\mu_2) \neq 0 \tag{3.1.21}$$

3.2 *Transformation to the Physical Plane:* The transformation from the σ-plane to the physical plane may be written

$$x = \int (E_1\varphi_\sigma\, d\sigma + \bar{E}_1\varphi_{\bar{\sigma}}\, d\bar{\sigma}) \tag{3.2.1}$$

$$y = \int (E_2\varphi_\sigma\, d\sigma + \bar{E}_2\varphi_{\bar{\sigma}}\, d\bar{\sigma}) \tag{3.2.2}$$

where E_1, E_2 are regular functions of $(\sigma^2, \bar{\sigma}^2)$. In view of this latter fact and the forms (3.1.13) and (3.1.15) it is clear that the T-periods (X_1, Y_1) and (X_2, Y_2) of φ_1 and φ_2 are

$$X_1 = Y_1 = 0, \qquad X_2 = Y_2 = 0 \tag{3.2.3}$$

since no logarithmic terms can arise from the integration. Moreover the T-periods (X_L, Y_L) and (X_M, Y_M) of φ_L, φ_M are, in view of (3.1.8) and (3.1.11)

$$\begin{cases} X_L = X^{(L,2)} = -T\sin\theta_0\,; & Y_L = Y^{(L,2)} = T\cos\theta_0 \\ X_M = X^{(L,1)} = -T\beta\cos\theta_0\,; & Y_M = Y^{(L,1)} = -T\beta\sin\theta_0 \end{cases} \tag{3.2.4}$$

where we use (2.2.1) and the fact that a closed curve surrounding $\sigma = 0$ in the σ-plane corresponds to a closed curve surrounding $\zeta = 0$ twice in the ζ-plane. Thus since

$$\begin{vmatrix} X_L & Y_L \\ X_M & Y_M \end{vmatrix} = T\beta \neq 0 \qquad \text{from (2.2.1a)} \tag{3.2.5}$$

the "consistency relations"

$$\begin{cases} \mu_1 X_1 + \mu_2 X_2 + \mu_L X_L + \mu_M X_M = 0 \\ \mu_1 Y_1 + \mu_2 Y_2 + \mu_L Y_L + \mu_M Y_M = 0 \end{cases} \tag{3.2.6}$$

reduce in this case to

$$\mu_L = \mu_M = 0 \qquad \mu_1, \mu_2 \text{ arbitrary} \tag{3.2.7}$$

This justifies the expression (3.1.20).

Inserting (3.1.20) into the transformation equations (3.2.1) and (3.2.2) we obtain

$$x = (ad/\sigma)[1 + \Sigma_0(\tilde{\sigma}^2)] + (\overline{ad}/\tilde{\sigma})[1 + \Sigma_0(\sigma^2)] + \Sigma(\sigma, \tilde{\sigma}) \tag{3.2.8}$$

$$y = (ad^*/\sigma)[1 + \Sigma_0(\tilde{\sigma}^2)] + (\overline{ad^*}/\tilde{\sigma})[1 + \Sigma_0(\sigma^2)] + \Sigma(\sigma, \tilde{\sigma}) \tag{3.2.9}$$

where d, d^* are given in (2.2.7). Defining new variables z, $\bar{z}$ as in (2.2.8) we obtain the expression corresponding to (2.2.9), namely:

$$z = (2a/q_0\sigma)[1 + \Sigma_0(\tilde{\sigma}^2)] + (1/\tilde{\sigma})\Sigma_0(\sigma^2) + \Sigma(\sigma, \tilde{\sigma}) \tag{3.2.10}$$

and similarly for $\bar{z}$.

3.3 *The Potential Function at Infinity:* In order to invert (3.2.10) we introduce two new variables τ, $\bar{\tau}$ which replace $\sigma^2/\tilde{\sigma}$, $\tilde{\sigma}^2/\sigma$ and four new functions v, $\bar{v}$, w, $\bar{w}$ which replace $\bar{z}/z^2$, $z/\bar{z}^2$, $1/z$, $1/\bar{z}$ respectively. This is done in an exactly similar way to that of 2.2. There is no need to introduce functions similar to ξ, $\bar{\xi}$ and u, $\bar{u}$ here since (3.2.10) contains no logarithmic terms. We then have (corresponding to (2.3.1) and (2.3.5)):

$$v = (q_0\bar{a}/2a^2)\tau[1 + \Sigma_0(\tau, \bar{\tau}, \sigma, \tilde{\sigma})] \tag{3.3.1}$$

$$w = (q_0/2a)\sigma[1 + \Sigma_0(\tau, \sigma, \tilde{\sigma})] \tag{3.3.2}$$

so that since

$$\frac{\partial(v, \bar{v}, w, \bar{w})}{\partial(\tau, \bar{\tau}, \sigma, \tilde{\sigma})} = \frac{q_0^4}{16a^2\bar{a}^2} \neq 0 \quad \text{at} \quad \tau = \bar{\tau} = \sigma = \tilde{\sigma} = 0 \tag{3.3.3}$$

we may invert (3.3.1) and (3.3.2) to obtain

$$\begin{cases} \tau = (2a^2/q_0\bar{a})v[1 + \Sigma_0(v, \bar{v}, w, \bar{w})] \\ \sigma = (2a/q_0)w[1 + \Sigma_0(v, \bar{v}, w, \bar{w})] \end{cases} \tag{3.3.4}$$

and similarly for $\bar{\tau}$, $\tilde{\sigma}$. A property of this inversion is that if we form from (2.3.4)

$$1/\sigma = (q_0/2aw)[1 + \Sigma_0(v, \bar{v}, w, \bar{w})] \tag{3.3.5}$$

then the series in the bracket contains no terms of the first order in v, $\bar{v}$, or $\bar{w}$.

The formulae (3.3.4) are valid for the particular value $\sigma^2/\tilde{\sigma}$ of τ. Hence for σ, $\tilde{\sigma}$ small enough we have inverted (3.2.10) to give:

$$\sigma = (2a/q_0 z)[1 + \textstyle\sum_{n=0}^{\infty} k_n(\alpha)/r^n] \tag{3.3.6}$$

where we have used (2.2.8). We note that $k_n(\alpha)$ and derivatives are continuous periodic functions of α. Substituting (3.3.6) into (3.1.20) and remembering the statement made about (3.3.5) we have

$$\varphi = q_0 r \cos(\alpha - \theta_0) + \textstyle\sum_{n=0}^{\infty} a_n(\alpha)/r^n \tag{3.3.7}$$

with $a_0(\alpha)$ independent of α. We can infer that the $a_n(\alpha)$ are continuous and periodic functions of α, together with their derivatives, and that (3.3.7) is uniformly and absolutely convergent for r large enough.

Again we may easily deduce that the series (3.3.7) can be differentiated term by term at will, and the resulting series have similar convergence properties.

4. Forces and Moments. 4.1 *Explicit form of coefficients:* It is possible to obtain the forces and moment on the system of immersed bodies as functionals of the singularities in the pseudo-logarithmic plane. This method is investigated by Bergman though his formulae are not interpreted physically. However, having established the expansions (1.1.2) and (1.1.3) it is more instructive to obtain them in the physical plane.

Without loss in generality we write $\theta_0 = 0$ and put $q_0 = U$. Moreover we write instead of (1.1.2):

$$\varphi/U = \cos\alpha + \bar{A}\log r + \bar{B} + \bar{C}(\log r/r) + \bar{D}/r + 0(\log^2 r/r^2) \tag{4.1.1}$$

where $\bar{A}$ is constant and $\bar{B}$, $\bar{C}$, $\bar{D}$ are functions of α. Using Bernoulli's equation to obtain the density in terms of $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$ . . . and using the density to obtain from (4.1.1) the outflow I across a control contour it may be shown that

$$I = 2\pi\bar{A}\beta U = m$$

where we equate the $m/2\pi\beta$ of (1.1.2) to the $\bar{A}U$ of (4.1.1). In order to simplify the analysis we shall take the case $m = 0$ which is of most interest. Then substituting (4.1.1) into the non-linear potential equation (ref. 3) in the physical plane, making use of Bernoulli's equation to evaluate the velocity of sound occurring in that equation, the following equations result for $\bar{B}$, $\bar{C}$, $\bar{D}$:

$$d^2\bar{B}/d\alpha^2 = M_0^2(d/d\alpha)[(d\bar{B}/d\alpha)\sin^2\alpha] \tag{4.1.2}$$

$$[(d^2/d\alpha^2) + 1][\bar{C}(1 - M_0^2\sin^2\alpha)] = 0 \tag{4.1.3}$$

$$\begin{aligned}[(d^2/d\alpha^2) + 1][\bar{D}(1 - M_0^2\sin^2\alpha)] \\ = -(\gamma-1)M_0^2\sin\alpha(d\bar{B}/d\alpha)(d^2\bar{B}/d\alpha^2) - (M_0^2/\sin\alpha)(d/d\alpha)[(d\bar{B}/d\alpha)^2\sin^2\alpha] \\ - 2M_0^2(d/d\alpha)[\bar{C}\sin\alpha\cos\alpha] + \bar{C}(2 - M_0^2)\end{aligned} \tag{4.1.4}$$

this equation holding also for the case $m \neq 0$.

The explicit solution of (4.1.2) is

$$\bar{B} = (\Gamma/2\pi U)\tan^{-1}(\beta\tan\alpha) \tag{4.1.5}$$

where we neglect an additive arbitrary constant and note that Γ is the value of the circulation, as can easily be verified. In passing we may state that if $\bar{A}$ is not taken zero, then $\bar{B}$ is

$$\bar{B} = (\Gamma/2\pi U)\tan^{-1}(\beta\tan\alpha) + \tfrac{1}{2}\bar{A}\log(1 - M_0^2\sin^2\alpha) \tag{4.1.6}$$

The additional term is periodic and in the notation of §2, §3 is $\frac{1}{2}\bar{A}\log(z\bar{z}/r^2)$ (compare (2.3.2)). The explicit solution of (4.1.3) is

$$\bar{C} = \frac{c\cos\alpha + c'\sin\alpha}{(1 - M_0^2\sin^2\alpha)} \qquad c, c' \text{ arbitrary.} \tag{4.1.7}$$

a term we should expect to arise (see (2.3.2)). Using (4.1.5) and (4.1.7) we integrate (4.1.4) to give

$$\begin{aligned}\bar{D}(1 - M_0^2 \sin^2 \alpha) &= B_1 \sin \alpha \tan^{-1}(\beta \tan \alpha) + B_2 \cos \alpha/(1 - M_0^2 \sin^2 \alpha) \\ &+ C_1 \sin \alpha \tan^{-1}(\beta \tan \alpha) + C_2 \cos \alpha \log (1 - M_0^2 \sin^2 \alpha) \\ &+ C_1' \cos \alpha \tan^{-1}(\beta \tan \alpha) + C_2' \sin \alpha \log (1 - M_0^2 \sin^2 \alpha) \\ &+ D_1 \cos \alpha + D_2 \sin \alpha\end{aligned} \tag{4.1.8}$$

where

$$B_1 = -(M_0^2/\beta)[1 + \tfrac{1}{4}(\gamma - 3)M_0^2](\Gamma/2\pi U)^2, \quad B_2 = \tfrac{1}{4}(\gamma + 1)M_0^2(\Gamma/2\pi U)^2$$

$$C_1 = \beta c; \quad C_1' = -(1/\beta)c'; \quad C_2 = \tfrac{1}{2}c; \quad C_2' = \tfrac{1}{2}c'; \quad D_1, D_2 \text{ arbitrary}$$

4.2 *Single-Valued Solution:* The equation (4.1.8) does not give a $\bar{D}$ which is periodic, together with its derivatives unless

$$B_1 = -C_1 \qquad C_1' = 0$$

These conditions imply that in (4.1.7)

$$\begin{cases} c = -(1/\beta)B_1 = (M_0^2/\beta^2)[1 + \tfrac{1}{4}(\gamma - 3)M_0^2](\Gamma/2\pi U)^2 = 0 \text{ only if } \Gamma = 0 \\ c' = 0 \end{cases}$$

Thus the constants in $\bar{C}$ are determined, not arbitrary. Since there is only a single infinity of arbitrary constants in (3.1.19) or (2.1.14), we may expect that all the constants of terms $a_{np}(\alpha)(\log r/r)^n(1/r^p)$ for $n \geq 1$ are determined by the periodicity condition, i.e. that these terms occur to retain periodicity in the $a_{0p}(\alpha)(1/r^p)$ terms. This is indicated in the formal method mentioned in (1.1). Hence $\bar{D}$ becomes

$$\bar{D} = \frac{B_2 \cos \alpha}{(1 - M_0^2 \sin^2 \alpha)^2} + \frac{C_2 \cos \alpha \log (1 - M_0^2 \sin^2 \alpha)}{(1 - M_0^2 \sin^2 \alpha)} + \frac{D_1 \cos \alpha + D_2 \sin \alpha}{(1 - M_0^2 \sin^2 \alpha)}$$

A simple computation using a control contour yields the force $\mathbf{F} = (F_x, F_y)$ on the system:

$$F_x = 0, \qquad F_y = \rho_0 U \Gamma \tag{4.2.1}$$

and the moment M about the origin

$$M = \rho_0 U^2 D_2 \tag{4.2.2}$$

The results (4.2.1) express the generalization of the Kutta-Joukowski formulae to the subsonic compressible case. It was first obtained by Glauert (ref. 4) without justification of the expansion for φ used. The result (4.2.2) is completely equivalent to the corresponding incompressible formula (ref. 1).

4.3 *Summary:* First the expansions (1.1.2) and (1.1.3), for the expansion at infinity of the potential function of a compressible flow, were justified. Then the explicit values of the first few terms in them were obtained. Finally the general-

izations to a compressible flow of the force and moment formulae for an incompressible flow were found.

We note here that if we wish to justify (1.1.3) with $a_1(\alpha) \equiv 0$, then one-third order singularities in the pseudo-logarithmic plane must be discussed, and so on. The analysis of §3 is only slightly modified.

We may note in conclusion that the expansions (1.1.2) and (1.1.3) are complete, in the sense that they cover all cases of the disturbance of a uniform stream. For if it were possible to have any other expansion, other types of singularity in the pseudo-logarithmic plane than those discussed in (2.1) and (3.1) would manifest themselves. However these other singularities will be excluded for the same reason that they are excluded in the incompressible case, see (1.2.1) and (1.2.2.); namely that the transformation to the physical plane (which has the same mathematical character in the two cases) will give multiple-valued flows in that plane.

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CIRCULAR CYLINDER AND FLAT PLATE AIRFOIL IN A FLOW FIELD WITH PARABOLIC VELOCITY DISTRIBUTION

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Introduction. In considering the problem of detached shock wave for a body at supersonic speeds (cf. Ref. 1), it was necessary to determine the effects of vorticity upon the velocity distribution in the subsonic region after a detached shock. With a curved detached shock wave the Mach number is subsonic back of the normal part of the wave and the Mach number after the shock wave becomes equal to the free stream Mach number at finite distance from the body. Thus, there is a region back of the detached shock wave and in front of the body in which the flow is subsonic, but it is also rotational because the curved shock wave causes the entropy to vary for the streamlines.

In order to obtain more information regarding the effects of subsonic rotational flow after the detached shock wave, an analysis was made for an incompressible flow with parabolic velocity distribution at infinity. The vorticity distribution in this case will be a linear function of y in the undisturbed region of the flow. In Ref. 2, Tsien has investigated the effect of linear velocity distribution, which corresponds to constant vorticity, upon the aerodynamic characteristics of the Joukowski Airfoil. In Ref. 3, Kuo has considered the same problem and used another method to derive the Blasius equations for flow with constant vorticity. Harrington and Libby (cf. Ref. 4) have considered the constant vorticity flow over a circular cylinder located near a rectilinear boundary.

The problem of flow with a parabolic type of velocity distribution is also encountered in flow through ducts. The effect of this type of flow upon the loads on objects in the duct can be investigated by applying the analysis that will be presented.

In the present paper, an analysis of the incompressible flow over a circular cylinder with and without circulation in a parabolic velocity distribution for the free stream is presented. The effects of parabolic and linear velocity distributions at infinity upon the flow over a flat plate airfoil at angles of attack are also considered.

A. Fundamental Equations and Assumptions. It will be assumed that the flow is incompressible, steady, non-viscous and two-dimensional. It will also be assumed that the velocity distribution at infinite distance ahead of the body is parabolic, namely,

$$u = U_0(1 + k(y^2/c^2)), \qquad v = 0 \tag{1}$$

where k is a non-dimensional constant, c is a representative length of the body and U_0 is the undisturbed flow velocity along the x-axis.

The equation of continuity for the two-dimensional, steady incompressible flow is

$$\partial u/\partial x + \partial v/\partial y = 0 \tag{2}$$

A stream function, ψ, which satisfies this equation is introduced and is defined by

$$u = \partial\psi/\partial y \qquad v = -\partial\psi/\partial x \tag{3}$$

The vorticity equation in terms of the stream function is then

$$-\Omega = \partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 \tag{4}$$

This vorticity is associated with the flow particle and maintains its strength; each streamline at infinite distance from the body will have a definite vorticity which remains constant along the streamline. Therefore, Eq. (4) may be written as

$$\nabla^2\psi = f(x) \tag{5}$$

Our problem is to solve this equation with the proper boundary conditions. The general solution of Eq. (5) is very difficult to obtain. We shall assume that the velocity curvature factor, k, is small and also that the disturbance due to the body is small such that the deflection of the streamline is small. Thus, we may write

$$\psi = \psi_0 + \psi_1 \tag{6}$$

where ψ_0 is the undisturbed stream function satisfying the equation

$$-\Omega_0 = \nabla^2\psi_0 = f(\psi_0) = (2U_0k/c^2)y \tag{7}$$

and ψ_1 is the disturbance stream function. The general vorticity equation now becomes

$$\nabla^2(\psi_0 + \psi_1) = f(\psi_0 + \psi_1) \tag{5a}$$

By Eqs. (7) and (5a), it is easily seen that

$$\nabla^2\psi_1 = f(\psi_0 + \psi_1) - f(\psi_0) \tag{8}$$

The right hand side of this last equation can be approximated by the use of derivative expression as follows

$$\nabla^2\psi_1 = [df(\psi_0)/d\psi_0]\psi_1 \tag{8a}$$

From Eqs. (1), (3), and (7),

$$\psi_0 = U_0[y+(ky^3/3c^2)] \tag{9}$$

and, then, we find

$$\begin{aligned} df(\psi_0)/d\psi_0 &= (2k/c^2)\ 1/[1 + k\ (y/c)^2] \\ &= (2k/c^2)[1 - k(y/c)^2 + k^2(y/c)^4 \cdots] \end{aligned} \tag{10}$$

provided that $k(y/c)^2 \ll 1.0$. Combination of Eqs. (8a) and (10) yields

$$\nabla^2\psi_1 = (2k/c^2)[1-k(y/c)^2 + \cdots]\psi_1 \tag{11}$$

Since it is assumed that k is small and also that the disturbance function ψ_1

is small, the product of these terms will be second order so may be neglected for the first approximation. Hence, from Eq. (11) we obtain the Laplace equation

$$\nabla^2\psi_1 = 0 \tag{12}$$

for the first approximation.*

B. Circular Cylinder in Shear Flow. Polar coordinates will be used in the following treatment. Thus, the normal and tangential components of the velocity vector will be defined respectively by

$$V_r = (1/r)\partial\psi/\partial\theta \qquad V_\theta = -\partial\psi/\partial r \tag{13}$$

If the circular cylinder is located at the origin as shown in Fig. 1, and the radius is taken to be $c/2$, the undisturbed velocity vector on the surface of the cylinder will be from Eq. (1),

$$u = U_0[1 + \tfrac{1}{4}k \sin^2\theta] \tag{14}$$

On the cylinder the normal component of the resultant velocity vector must be zero; this condition is analytically expressed as

$$[(1/r)\partial\psi_1/\partial\theta]_{r=c/2} = -u\cos\theta = -U_0[(1 + \tfrac{1}{16}k)\cos\theta - \tfrac{1}{16}k\cos 3\theta] \tag{15}$$

We are therefore looking for a solution of Eq. (12) which satisfies the boundary condition in Eq. (15) on the cylinder and vanishes at infinite distance away from the cylinder. Such a solution of the Laplace equation for ψ_1 may be written as

$$\psi_1 = U_0[-\tfrac{1}{4}c^2(1 + \tfrac{1}{16}k)r^{-1}\sin\theta + \tfrac{1}{3}k(\tfrac{1}{256}c^4)r^{-3}\sin 3\theta] \tag{16}$$

Hence, by Eqs. (9) and (16), the resultant stream function is

$$\begin{aligned}\psi = \psi_0 + \psi_1 = U_0\{&[r - \tfrac{1}{4}c^2(1 + \tfrac{1}{16}k)r^{-1}]\sin\theta \\ &+ \tfrac{1}{3}k[(r^3/c^2)\sin^3\theta + \tfrac{1}{256}c^4r^{-3}\sin 3\theta]\end{aligned} \tag{17}$$

which is similar to the expression obtained by Tsien in Ref. 2 for constant vorticity distribution. The tangential velocity on the cylinder is, from Eq. (17),

$$\begin{aligned}(V_\theta)_{r=c/2} &= -(\partial\psi/\partial r)_{r=c/2} \\ &= -U_0\{(2 + \tfrac{1}{16}k)\sin\theta + \tfrac{1}{4}k\,(\sin^3\theta - \tfrac{1}{4}\sin 3\theta)\}\end{aligned} \tag{18}$$

Thus, the stagnation points occur at $\theta_s = 0$ and π for all values of k. This, of course, should be expected because the parabolic velocity distribution at $x = -\infty$ is symmetrical about the x-axis. The maximum velocity on the surface of the cylinder occurs at $\theta = \pm\pi/2$ which correspond to the intersections of the cylinder with the y-axis and are the same locations as for uniform flow without vorticity. The magnitude for this maximum velocity is

$$(V_{\theta_{max}})_{r=c/2} = -U_0(2 + \tfrac{3}{8}k) \tag{19}$$

* The second approximation for the disturbance stream function is obtained by considering Eq. (11) with the first term on the right hand side and the solution is expressed in terms of the modified Bessel functions.

In Eqs. (18) and (19), set $k = 0$, we obtain respectively

$$(V_\theta)_{r=c/2} = -2U_0 \sin\theta \tag{18a}$$

$$(V_{\theta_{max}})_{r=c/2} = -2U_0 \tag{19a}$$

which are, of course, the correct answers for the case of uniform flow at infinity.

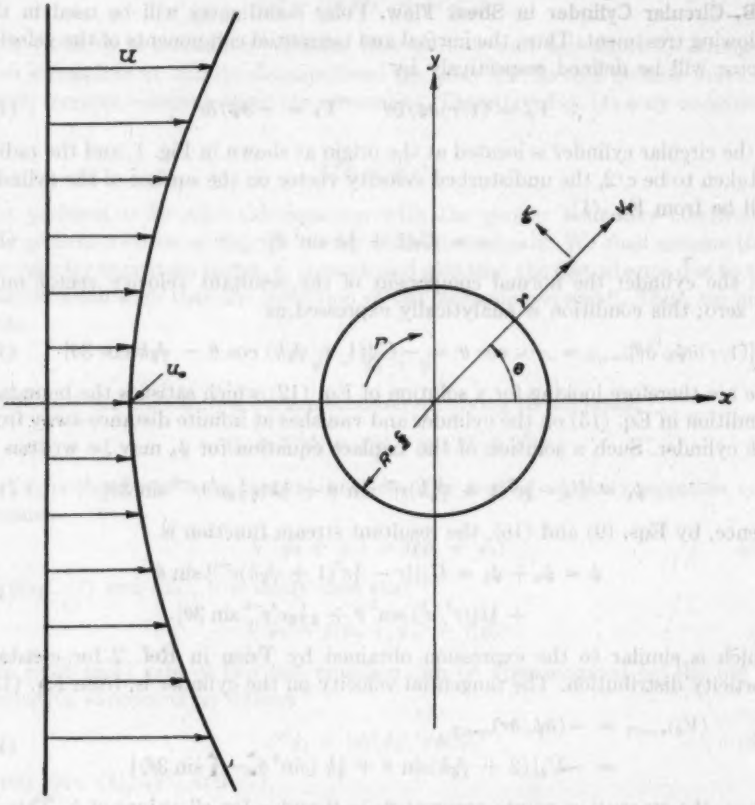

FIG. 1. Circular Cylinder in Shear Flow.

In considering the detached shock wave, it is important to determine the effect of vorticity upon the velocity distribution along the negative x-axis. Only the radial velocity exists along the x-axis since it is the axis of symmetry. The radial velocity at any point along the negative x-axis, $\theta = \pi$, is given by

$$(V_r)_{\theta=\pi} = U_0\{-[1 - \tfrac{1}{4}c^2(1 + \tfrac{1}{16}k)\, r^{-2}] - \tfrac{1}{256}kc^4r^{-4}\} \tag{20}$$

If we let $k = 0$ in this equation, the result is

$$(V_r)_{\theta=\pi} = -U_0[1 - (c^2/4r^2)] \tag{20a}$$

which is the correct expression for the flow without vorticity. For a given ratio of $(V_{r_{\theta=\pi}}/U_0)$, the location on the negative x-axis of this velocity ratio will be farther away from the origin for the rotational flow than for the uniform velocity as can be seen by comparing Eqs. (20) and (20a). In other words, the effect of vorticity has made the location for a given decrease in free stream velocity farther away than for the case of uniform flow.

C. Circular Cylinder With Circulation in Flow With Vorticity. Consider the case of a circular cylinder with circulation in an incompressible flow with parabolic velocity distribution at $x = -\infty$ as indicated in Fig. 1. The resultant first approximation stream function is then

$$\psi = U_0\left\{\left[r - \frac{c^2}{4}\left(1 + \frac{k}{16}\right)\frac{1}{r}\right]\sin\theta + \frac{k}{3}\left[\frac{r^3}{c^2}\sin^3\theta + \frac{c^4}{256}\frac{\sin 3\theta}{r^3}\right]\right\} + \frac{\Gamma}{2\pi}\log r \tag{21}$$

where Γ is the strength of the circulation and will be considered positive for the clockwise direction. Clearly, this function ψ satisfies the boundary condition in Eq. (15). For this case, the velocity on the surface of the cylinder is

$$(V_\theta)_{r=c/2} = -U_0\{(2 + \tfrac{1}{16}k)\sin\theta + \tfrac{1}{4}k(\sin^3\theta - \tfrac{1}{4}\sin 3\theta)\} - \Gamma/\pi c \tag{22}$$

which differs from the expression in Eq. (18) only by the last term due to the circulation. The stagnation points on the cylinder can be determined by setting Eq. (22) equal to zero and solving for θ_s. Thus, we have

$$\begin{aligned}\theta_{s_1} &= \sin^{-1}[\tfrac{1}{3}(A + B - b)]\\ \theta_{s_2} &= \sin^{-1}[\tfrac{1}{3}(\omega^2 A + \omega B - b)]\\ \theta_{s_3} &= \sin^{-1}[\tfrac{1}{3}(\omega A + \omega^2 B - b)]\end{aligned} \tag{23}$$

where

$$A^3 = \frac{1}{2}\left\{\frac{-54\Gamma}{\pi c U_0 k} + \left[\left(\frac{54\Gamma}{\pi c U_0 k}\right)^2 + 108\left(\frac{4}{k} - \frac{1}{4}\right)^3\right]^{\frac{1}{2}}\right\}$$

$$B^3 = \frac{1}{2}\left\{\frac{-54\Gamma}{\pi c U_0 k} - \left[\left(\frac{54\Gamma}{\pi c U_0 k}\right)^2 + 108\left(\frac{4}{k} - \frac{1}{4}\right)^3\right]^{\frac{1}{2}}\right\}, \qquad \omega = \tfrac{1}{2}(-1 + i\sqrt{3})$$

Hence, it is seen that the location of the stagnation point depends upon the vorticity k, circulation Γ and the velocity U_0.

The Bernoulli equation for incompressible flow is

$$P_0 + \tfrac{1}{2}\rho_0 U_0^2 = P + \tfrac{1}{2}\rho_0 (V_\theta)^2_{r=c/2} \tag{24}$$

Therefore, the pressure coefficient on the cylinder surface is

$$\begin{aligned}c_p = \frac{P - P_0}{\frac{1}{2}\rho_0 U_0^2} &= 1 - \left(\frac{V_{\theta_{r=c/2}}}{U_0}\right)^2\\ &= 1 - \left\{\left[\left(2 + \frac{k}{16}\right)\sin\theta + \frac{k}{4}\left(\sin^3\theta - \frac{\sin 3\theta}{4}\right)\right] + \frac{\Gamma}{\pi c U_0}\right\}^2\end{aligned} \tag{25}$$

This equation shows that the effect of vorticity, k, for the same circulation strength is to make the pressure coefficient more negative than for the flow without vorticity.

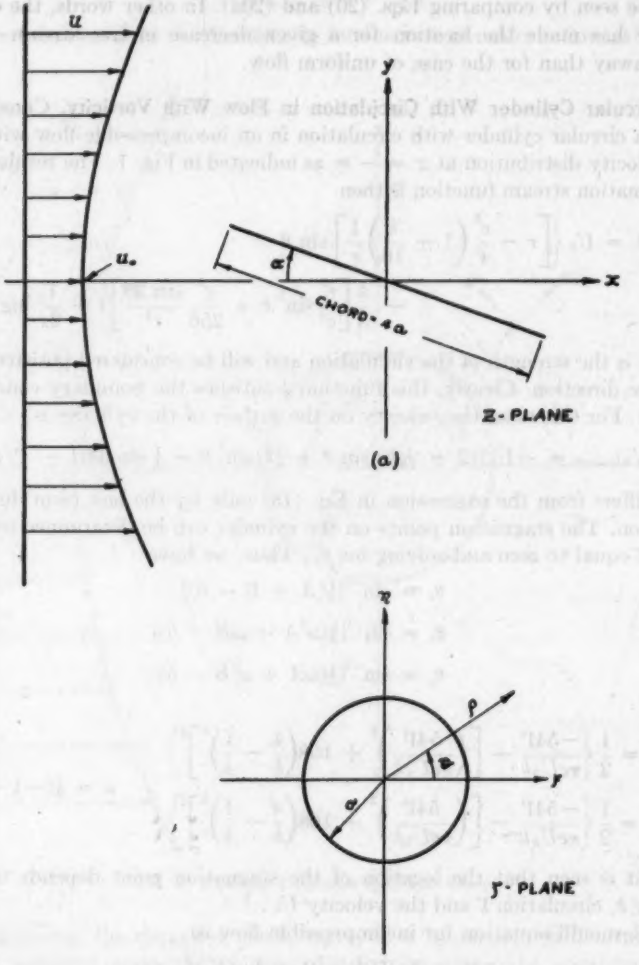

FIG. 2. Flat Plate Airfoil in Shear Flow.

D. Flat Plate Airfoil in Flow With Vorticity. 1. Flat Plate Airfoil in a Parabolic Velocity Distribution at Infinity.

A flat plate airfoil in a flow with a parabolic velocity distribution at infinity as shown in Fig. 2 is considered in this section. As shown in Eq. (12), for the

first approximation the disturbance function ψ_1 satisfies the Laplace equation, and thus, the method of conformal transformation will be utilized to analyze this function. Conformal transformation cannot be applied to ψ_0 which does not satisfy the Laplace equation. The Joukowski transformation will be used to transform the circle of radius a in the ζ-plane into a flat plate of chord $= 4a$ in the z-plane at an angle of attack of α as shown in Fig. 2. The transformation equation is

$$z = e^{-i\alpha}(\zeta + a^2/\zeta) \qquad \zeta = \rho e^{i\theta} \tag{26}$$

Thus, the values of x and y corresponding to a point on the surface of the circle, $\rho = a = c/2$, in the ζ-plane are given as follows:

$$x = c \cos\theta \cos\alpha, \qquad y = -c \cos\theta \sin\alpha \tag{27}$$

The boundary condition of no normal flow across the airfoil must be used to determine the first approximation disturbance stream function ψ_1. By Eqs. (1) and (27), it is seen that

$$u = U_0\{1 + k \cos^2\theta \sin^2\alpha\} \tag{28}$$

Then our boundary condition on the surface of the airfoil can be expressed analytically as

$$(1/U_0)(\rho^{-1}\partial\psi_1/\partial\theta)_{\rho=a} = -2 \sin\theta \sin\alpha\,(1 + k \cos^2\theta \sin^2\alpha) \tag{29}$$

Besides Eq. (29), it is necessary to use the Kutta-Joukowski condition at the trailing edge of the airfoil. The strength of the circulation is determined by the Kutta-Joukowski condition which states that the velocity at the trailing edge of the airfoil must be finite. Since the normal component of the velocity must be zero to satisfy the boundary condition Eq. (29), only the tangential component of the velocity exists on the surface of the airfoil. The resultant tangential velocity on the airfoil surface can be expressed as

$$V_{\tau_R} = [(\partial\psi_1/\partial\rho)_{\rho=a} + 2\,u \cos\alpha \sin\theta](2 \sin\theta)^{-1} \tag{30}$$

At the trailing edge, where $\rho = a$ and $\theta = 0$, the quantity within the bracket of Eq. (30) must vanish in order to have finite velocity at the trailing edge. Thus, we have by Eq. (28)

$$(1/U_0)(\partial\psi_1/\partial\rho)_{\rho=a} = -2(1 + k \cos^2\theta \sin^2\alpha) \cos\alpha \sin\theta \qquad \text{at } \theta = 0 \tag{31}$$

It can be found without difficulty that the solution ψ_1 which satisfies these conditions, Eqs. (29) and (31), is

$$\psi_1 = U_0\{a[2 \sin\alpha + k \sin^3\alpha] \log\rho + 2a^2 \sin\alpha\,(1 + \tfrac{1}{4}k \sin^2\alpha)\rho^{-1}\cos\theta + \tfrac{1}{4}ka^4 \sin^3\alpha\rho^{-3} \cos 3\theta\} \tag{32}$$

By Eqs. (30) and (32), the resultant tangential velocity on the airfoil is found after some reduction, to be

$$V_{\tau_R} = (U_0/\sin\theta)[\sin(\theta - \alpha) + \sin\alpha + \tfrac{9}{4}k \sin^2\alpha (4 \cos^2\theta \sin\theta \cos\alpha + 2 \sin\alpha - \sin\alpha \cos\theta - \sin\alpha \cos 3\theta)] \tag{33}$$

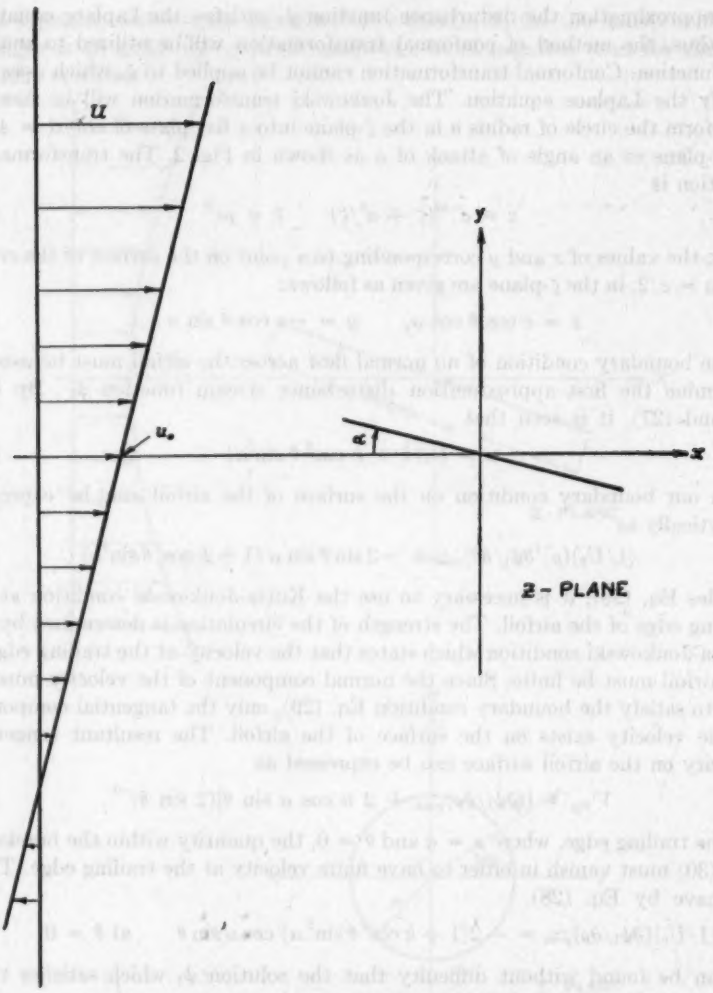

FIG. 3. Flat Plate Airfoil in Constant Shear Flow.

If $k = 0$, Eq. (33) reduces to

$$V_{r_2} = (U_0/\sin\theta)\{\sin(\theta - \alpha) + \sin\alpha\} \tag{34}$$

which is the velocity on a flat plate airfoil in a uniform flow without vorticity. Thus, the effect of vorticity is to make the maximum velocity on the upper surface at a particular angle of attack greater than that for the case without vorticity at the same angle of attack and velocity, U_0. Also, it can be seen from

Eq. (32) that the strength of the circulation also increases with the vorticity of the flow for a given angle of attack.

2. Flat Plate Airfoil in a Constant Vorticity Flow.

For a constant vorticity flow, the velocity at infinite distance upstream from the airfoil will vary linearly with y as follows (cf. Fig. 3).

$$u = U_0(1 + ky/c) \tag{35}$$

The corresponding undisturbed stream function is

$$\psi_0 = U_0[y + (k/2c)y^2] \tag{36}$$

By Eq. (27), on the surface of the airfoil, velocity u in Eq. (35) becomes

$$u = U_0(1 - k \cos\theta \sin\alpha) \tag{35a}$$

As in Eq. (29), the boundary condition of no flow across the airfoil is expressed as

$$(1/U_0)(\rho^{-1}\partial\psi_1/\partial\theta)_{\rho=a} = -2 \sin\theta \sin\alpha(1 - k \cos\theta \sin\alpha) \tag{37}$$

And, as in Eq. (31) the Kutta-Joukowski condition is expressed as

$$(1/U_0)(\partial\psi_1/\partial\rho)_{\rho=a} = -2 \cos\alpha \sin\theta(1 - k \cos\theta \sin\alpha), \qquad \text{at } \theta = 0 \tag{38}$$

ψ_1 is thus found to be

$$\psi_1 = U_0\left\{2a \sin\alpha\left(1 - \frac{k}{2}\sin\alpha\right)\log\rho + \frac{2a^2 \sin\alpha \cos\theta}{\rho} - \frac{ka^3}{2}\frac{\sin^2\alpha \cos 2\theta}{\rho^2}\right\} \tag{39}$$

By Eqs. (30), (35a), and (39) the resultant tangential velocity on the airfoil is

$$V_{T_R} = (U_0/\sin\theta)\{\sin(\theta - \alpha) + \sin\alpha - \tfrac{1}{2}k \sin\alpha (\sin^2\theta \cos\alpha + \sin\alpha - \sin\alpha \cos 2\theta)\} \tag{40}$$

If $k = 0$, this again reduces to Eq. (34). Thus, the effect of constant vorticity is to decrease the magnitude of the maximum velocity on the upper surface for a given angle of attack. Also, from Eq. (39), it is seen that the circulation decreases with the amount of constant vorticity. These tendencies are of opposite signs in comparison with the previous case of parabolic velocity distribution.

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ON A RELAXATION METHOD FOR EIGENVALUE PROBLEMS

By Stephen H. Crandall

A Relaxation method is essentially a method of successive approximations in which the computation is so arranged that the error corresponding to a given trial solution is readily found and moreover is of sufficiently suggestive form to guide the computer in altering the given trial in the right direction. R. V. Southwell and others[1] have shown a Relaxation method for finding the roots and vectors of matrices. They have also discussed the eigenvalue problem for two-dimensional continua[2] by using finite difference approximations to the governing differential equation. A. Vazsonyi[3] has given an alternant Relaxation method for this latter class of problem. The purpose of this paper is to point out that Vazsonyi's technique[4] is particularly well adapted to finding rough approximations to the roots and vectors of matrices and to give a proof of the convergence of the method. The procedure has the advantage that the same computational machinery is used for all modes and that orthogonalization of trial vectors is not required.

1. Description of the Method. Let A be a real symmetric matrix of order n with elements a_{ij}. Then the eigenvalue problem is to find vectors $X(x_1 \cdots x_n)$ and corresponding scalars λ which satisfy $AX = \lambda X$, or

$$\sum_j a_{ij}x_j = \lambda x_i, \qquad (i = 1, \cdots, n). \quad (1)$$

We note in passing that this formulation includes the case treated by Vazsonyi. The method of Vazsonyi when applied to this problem consists in associating a set of numbers, $l_i (i = 1, \cdots, n)$, with a trial vector $V(v_1 \cdots v_n)$ by the equations

$$\sum_j a_{ij}v_j = l_i v_i, \qquad (i = 1, \cdots, n) \quad (2)$$

and successively correcting the trial vector with intent to equalize the l_i. There are two properties[5] of the l_i which facilitate the computation; (1) the enclosure property, which insures that if the algebraicly least and greatest l_i are $l_{\min}$ and $l_{\max}$ respectively then there is always at least one eigenvalue, λ, satisfying

$$l_{\min} \leq \lambda \leq l_{\max} \quad (3)$$

and (2) the fact that Rayleigh's quotient, λ_R, is a weighted average of the l_i;

$$\lambda_R = \frac{\sum_i \sum_j a_{ij}v_i v_j}{\sum_i v_i^2} = \frac{\sum_i l_i v_i^2}{\sum_i v_i^2}. \quad (4)$$

[1] Southwell, R. V., "Relaxation Methods in Engineering Science," Oxford University Press, London, 1940.

[2] Southwell, R. V., "Relaxation Methods in Theoretical Physics," Oxford University Press, London, 1946.

[3] Vazsonyi, A., A Numerical Method in the Theory of Vibrating Bodies, *J. Appl. Phys.* **15**, 598–606 (1944).

[4] A similar technique appears on p. 292 of L. Collatz, "Eigenwertaufgaben mit Technischen Andwendung," Akad. Verlags., Leipzig, 1949. No discussion of convergence is given.

[5] See, for example, the reference of footnote (4).

The actual arrangement of the calculation utilizes a technique made familiar by Southwell, namely the use of a unit operations table to facilitate the rapid evaluation of the effect of alterations to an existing trial vector. For each of the components we tabulate three quantities: the current value of v_i, the current value of the left side of (2), and the quotient of these two which according to the right side of (2) yields l_i. As we alter the v_i we are able to write down the alterations in the left sides of (2) with the help of the unit operations table. For exploratory calculations the quotients may be obtained quickly with a slide rule. To illustrate the technique we shall consider a numerical example before passing on to a more rigorous examination.

TABLE I

	v_1	l_{1v_1}	l_1	v_2	l_{2v_2}	l_2	v_3	l_{3v_3}	l_3	v_4	l_{4v_4}	l_4
(1)	100	200	2.0	100	100	1.0	100	0	0	100	200	2.0
(2)		150	1.5	150	250	1.7		50	0.5		100	1.0
(3)					225	1.5	75	75	1.0			
(4)	100	0	0	−100	−100	1.0	100	−200	−2.0	−100	200	−2.0
(5)					100	−1.0	300	−400	−1.3			
(6)	50	−50	−1.0		150	−1.5					100	−1.0
(7)	100	400	4.0	−100	−700	7.0	−100	0	0	100	600	6.0
(8)					−615	6.2	−15	−85	5.7			
(9)	100	−200	−2.0	100	400	4.0	0	100	∞	−100	−200	2.0
(10)					300	3.0	−100	200	−2.0			
(11)		−100	−1.0	0	0	0/0		100	−1.0		0	0
(12)					50	∞	−50	50	−1.0			
(13)		−85	−0.9	−15	5	−0.3		35	−0.7		30	−0.3

2. Example. Let us find the roots and vectors of the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -1 & 3 & 1 & -2 \\ 0 & 1 & -1 & 0 \\ 2 & -2 & 0 & 2 \end{bmatrix}$$

For simplicity we will consider only the four unit operations of adding unity to $v_i (i = 1, 2, 3, 4)$. Then corresponding additions to the left sides of (2) are given by the elements in the i-th column of A. Table I shows a sample computation beginning in line (1) with a trial vector having components (100, 100, 100, 100). The left sides of (2) are computed and entered in the second columns as 200, 100, 0, and 200 respectively. The l_i are then obtained by dividing the first columns into the second columns to get the set of $l_i(2, 1, 0, 2)$. We can thus be assured that at least one eigenvalue of A is included between 0 and 2. Two steps of the Relaxation are shown: an addition of 50 to v_2 in line (2) and a subtraction of 25 from v_3 in line (3) which results in the set of $l_i(1.5, 1.5, 1.0, 1.0)$.

An important characteristic of the Relaxation process is that these operations have been of an exploratory nature. In no sense are they best possible operations. The operations were tried and since they resulted in a decreased spread of l_i, they were allowed to stand. To illustrate how the method is used to locate other eigenvalues we continue by starting anew with the trial vector (100, -100, 100, -100) in line (4). This vector was chosen as being (very crudely) approximately orthogonal to the vector of line (3). The operations of lines (5) and (6) are intended to narrow the spread of the l_i without enclosing the range of l_i already obtained in line (3). After adding 200 to v_3 in line (5) and -50 to v_1 in line (6) we find the set of $l_i(-1.0, -1.5, -1.3, -1.0)$. In line (7) a new trial vector (100, -100, -100, 100) yields the set of $l_i(4, 7, 0, 6)$ which includes the set already found in line (3) but it requires only the single alteration of an addition of 85 to v_3 in line (8) to isolate another eigenvalue between 4.0 and 6.2. With three of the eigenvalues of A roughly located we seek after the fourth in line (9) with the trial vector (100, 100, 0, -100). The resulting set of l_i $(-2, 4, \infty, 2)$ unfortunately includes all three of the earlier sets of lines (3), (6), and (8). Lines (10) and (11) illustrate exploratory alterations in an attempt to find a set of l_i which does not include any of the earlier sets. The sets of line (11) and line (6) while not overlapping do have the value -1.0 in common. To insure that we really have separated two modes we carry on the Relaxation process two more steps after line (11) to obtain the set of $l_i(-0.9, -0.3, -0.7, -0.3)$ in line (13).

At the present stage of the calculation we have rough approximations to the four eigenvectors and have the following bounds for the eigenvalues; $-1.5 \leq \lambda_1 \leq -1.0$, $-0.9 \leq \lambda_2 \leq -0.3$, $1.0 \leq \lambda_3 \leq 1.5$ and $4.0 \leq \lambda_4 \leq 6.0$. After carrying out five additional steps on each mode these bounds were improved to; $-1.34 \leq \lambda_1 \leq -1.32$, $-0.61 \leq \lambda_2 \leq -0.60$, $1.35 \leq \lambda_3 \leq 1.36$ and $5.55 \leq \lambda_4 \leq 5.58$. It is considered that this represents about the useful limit of the accuracy of this method. Greater accuracy would require more accurate quotients than are convenient for the slide rule, or else an alteration of the computation such as that suggested by Vazsonyi[3]. It is felt, however, that the chief merit of this method lies in its simplicity and speed in the early exploratory stages of an eigenvalue calculation.

In common with other Relaxation procedures no definite rule is laid down for choosing the alterations at each stage other than the basic principle that an alteration has been useful if the new set of l_i more nearly approach the condition of equality. Because of (4) we see that it is more important to equalize the l_i corresponding to the larger v_i. The l_i corresponding to the smaller v_i are readily brought into line afterwards. See line (13) of Table I. It may be helpful when the computations have reached a stage comparable with that of Table I, to construct auxiliary tables for each mode giving

$$\partial l_i/\partial v_j = (a_{ij} - \delta_{ij} l_i)/v_i \qquad (5)$$

(δ_{ij} is the Kronecker δ) which could be used to facilitate the selection of alterations to the v_j in order to obtain desired alterations of the l_i.

When k of the modes have been determined and it is required to obtain another mode it is only necessary to try vectors until one is found which gives a set of l_i which does not include any of the k known eigenvalues. The process of finding this new trial vector is illustrated in the above example by lines (9), (10) and (11) of Table I. When such a vector has been found the procedure once more consists in selecting alterations which reduce the spread of the l_i. It is helpful for the computer to keep in mind that a new eigenvector will be orthogonal to all the previously determined eigenvectors. This fact should be used as a rough guide (but only as a rough guide) in selecting new trial vectors; e.g., the sign pattern of the elements of the trial vector should differ from that of the known eigenvectors. It is actually uneconomic to exactly orthogonalize a trial vector in this procedure since it is easily possible to have trial vectors orthogonal to k known eigenvectors, which give sets of l_i for which all of the k known eigenvalues are included between $l_{\min}$ and $l_{\max}$.

3. Convergence. The freedom in choice of operation which is given a computer in a Relaxation method makes the concept of convergence for such procedures somewhat nebulous. It is often possible to construct an iterative process within the framework of a Relaxation scheme by selecting a certain single criterion which unambiguously determines each succeeding operation. We propose to call a Relaxation process convergent if there exists such a derived iterative process which is itself convergent. A convergent iterative process derived from the Relaxation scheme outlined in this paper will now be developed. It should be made clear that this iterative procedure is not suggested as being an attractive one for computation. With the computation scheme described above most computers could stumble on solutions by trial and error much faster than by adopting the following procedure. The point is merely to show that *there does exist* a systematic procedure.

The basic idea is to show that at any stage of the process an alteration of the v_j can be found which will decrease the spread of l_i and to obtain a quantitative measure of the improvement. This is essentially equivalent to giving a constructive proof of the enclosure property.

For a given V, eq. (2) uniquely determines a set of l_i. On the other hand a set of l_i determine a non-null V only if the rank of the matrix

$$B = \begin{bmatrix} a_{11} - l_1 & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} - l_2 & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} - l_n \end{bmatrix} \tag{6}$$

is less than n. We will call such a set of l_i compatible. If the rank of B is $n - 1$ a unique vector direction is determined; i.e., a vector V is uniquely determined to within a multiplicative factor. If the rank of B is $n - r$ we may say that r orthogonal directions are determined. In the iterative scheme to be described we will fix our attention on compatible sets of l_i and will construct a fixed procedure for passing from one such set to another. At any stage a corresponding V may be

determined if desired by solving eq. (2) for V. As a quantitative measure of the error associated with a set of l_i we define the error function

$$E = \sum_i (l_i - M)^2 \tag{7}$$

where M is the current arithmetic mean of the l_i.

In each step of the iterative procedure we start from a vector V, a set of l_i, and an error E and replace these with V', l_i' and E' according to the following plan. We single out two particular l_i, namely l_p and l_q, such that $l_p = l_{\min}$ and $l_q = l_{\max}$. Then the l_i' are a compatible set of l_i such that $l_i' = l_i$ for $i \neq p, q$ but with either $l_p' > l_{\min}$ or $l_q' < l_{\max}$, or both, and moreover with $E' < E$. We proceed to show that such a set of l_i' always exists and that only a finite number of such steps are necessary to make E less than any preassigned positive ϵ. This is equivalent to saying that the l_i have a common limit λ, which is an eigenvalue of A. This also constitutes a direct proof of the enclosure property.

Let the determinant of the matrix B be denoted by D. Consider l_p and l_q to be differentiable functions of a parameter t, while for $i \neq p, q$ the l_i remain constant. Then if the set of l_i is to remain compatible for continuous change of the parameter t, we must have

$$(\partial D/\partial l_p)(dl_p/dt) + (\partial D/\partial l_q)(dl_q/dt) = 0. \tag{8}$$

Now $(\partial D/\partial l_p)$ and $(\partial D/\partial l_q)$ are proportional to main diagonal cofactors of a singular determinant and hence always have the same sign. Postponing, for the moment, the exceptional cases where one or both of $(\partial D/\partial l_p)$ and $(\partial D/\partial l_q)$ are zero, we see that to satisfy (8) increasing l_p must be accompanied by decreasing l_q. Now

$$\tfrac{1}{2}dE/dt = (l_q - M)(dl_q/dt) - (M - l_p)(dl_p/dt) \tag{9}$$

which is clearly negative for increasing l_p and decreasing l_q at least so long as $l_p < M < l_q$. To make the determination of the set l_i' definite, we consider that starting from the original l_i, l_p and l_q are continuously increased and decreased according to (8) until, using current values, $(l_q - M)(M - l_p)$ first vanishes. The set of l_i so determined is called l_i'. If, for instance, $\partial D/\partial l_p = 0$ the set l_i' is determined by increasing l_p with l_q held constant until, using current values, $M - l_p$ first vanishes. It may be easily verified from (7) that

$$\begin{aligned} E - E' = n(M' - M)^2 &+ (M' - l_p)^2 \\ &+ (l_q - M')^2 - (M' - l_p')^2 - (l_q' - M')^2 \end{aligned} \tag{10}$$

where M' and E' are the mean and error for the set l_i'. Let $l_{\max} - l_{\min}$ be denoted by h. A study of (10) reveals that $E - E'$ will always be greater than $(h/n)^2$. For suppose $l_p' = M'$ then either $l_p' - l_p \geq l_q - l_q'$ or $l_p' - l_p < l_q - l_q'$. In the first case the third and fifth terms of (10) taken together are non-negative as is the first term while $M' - l_p$ is certainly greater than h/n. On the other hand if $l_p' - l_p < l_q - l_q'$ we have the first and second terms of (10) non-negative and the difference between the third and fifth terms is clearly greater than $(h/n)^2$ since $l_q - l_q'$ must now be greater than h/n. A similar argument holds if instead,

it is l_q' that equals M'. Thus for any compatible set of l_i we have indicated a procedure for determining a compatible set l_i' such that $E - E' > (h/n)^2$. If we now combine with this the obvious fact that $E < nh^2$ we have

$$E'/E < 1 - (1/n^3) \tag{11}$$

Hence at the end of k steps in this process the error function will be less than $(1 - 1/n^3)^k$ times the initial error function which can be made smaller than any preassigned positive ϵ if k is taken large enough. Thus there exists within the framework of the Relaxation scheme a fixed procedure which converges to an eigenvalue of (1) enclosed between the $l_{\min}$ and $l_{\max}$ of each set of l_i of the sequence.

4. Concluding Remarks. We can show that there exists a neighborhood about the point $l_1 = l_2 = \cdots = l_n = \lambda_s$ for which if $l_i = \lambda_s + \epsilon_i$ with $|\epsilon_i| < \epsilon$, then $\lambda_R = \lambda_s + 0(\epsilon^2)$. If X and V are normalized vectors satisfying (12) and (13) respectively

$$a_{ij}x_j = \lambda_s x_i \tag{12}$$

$$a_{ij}v_j = l_i v_i \tag{13}$$

and if $d = \min |\lambda_s - \lambda_r|$, $(r \neq s)$, then it is known[6,7] that

$$|X - V| = 0(\epsilon/d) \tag{14}$$

and hence

$$\lambda_R = \lambda_s + 0(\epsilon^2/d^2) \tag{15}$$

using the well-known property of Rayleigh's quotient. It may be remarked that in the other direction, the fact that $|X - V|$ is small does not insure that the ϵ_i defined above will be small. In fact if X has any zero elements the corresponding ϵ_i will be unbounded.

The Relaxation procedure described herein can be extended immediately to the eigenvalue problem

$$AX = \lambda BX \tag{16}$$

if B is a diagonal matrix of positive elements. However it should be noticed that if B is simply a matrix of a positive definite form then the enclosure property can no longer be guaranteed and the method loses its intuitive simplicity.

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[6] See Satz V, H. WITTMEYER, Einfluss der Anderung einer Matrix auf die Losungen des Zugehorigen Gleichungssystems, sowie auf die Characterischen Zahlen und die Eigenvektoren, *Z. ang. Math. und Mech.* **16**, 287-300 (1936).

[7] If λ_s is a multiple root, there is always some X satisfying (12) and (14).

THE DETERMINATION OF POSITION AND VELOCITY ON THE EARTH'S SURFACE

By Charles Fox

1. Introduction. The aim of closed navigation is to determine position and velocity on the earth's surface by means of observations or measurements which can be made inside a closed room. For example the determination of latitude and longitude by the measurement of two stellar altitudes is excluded from the operations of closed navigation.

In this discussion we assume that the body carrying the apparatus to be described later is sufficiently small in relation to the earth to be represented by a point P. P's latitude and longitude, at any instant, will be denoted by λ and ϕ respectively, and its northerly and westerly components of velocity by n and w respectively. These four quantities are not all independent of each other, for evidently $\dot{\lambda} = n/a$ and $\dot{\phi} = w/(a \cos \lambda)$, where a denotes the radius of the earth. The problem of closed navigation, stated more precisely, is to determine λ, n, w and a definite direction, such as local true north, by means of measurements or observations carried out inside a closed room.

ϕ cannot be found by closed navigation alone since its zero position depends upon the arbitrary choice of a meridian. But it will appear later that we can find w and consequently can integrate $w/(a \cos \lambda)$ with respect to time and so determine ϕ, except for an arbitrary constant.

I have already given a solution of the problem of closed navigation for slowly moving bodies such as vessels at sea. It appears in a paper entitled* "The Mechanical Determination of Position and Velocity on the Earth's Surface." In that solution the use of the gyrocompass was fundamental but in the one given here this instrument is not needed at all. The direction obtained by the gyrocompass is found here by a different method. It is also possible that the alternative solution proposed here can be put onto practical form with greater ease than my previous one.

The discussion below, which is based upon properties of gyroscopes, is entirely theoretical, since much experiment would be required before the ideas involved could be put to practical use.

2. Apparatus Required. We must first fix the vertical at P. This can be done with accuracy and stability by means of either gravity or pendulum controlled gyroscopes. It will be assumed henceforth that the vertical is known.

The plane of figure 1 is supposed to be horizontal. A gyroscope is mounted so that its spin axis AA' can freely rotate about the vertical at P. The gyroscopic wheel is shown in the figure by its rectangular horizontal cross section with centre at P. The spin of the wheel about APA' will be denoted by s, radians per second, and s will be positive if the spin is clockwise on looking from P towards A' and otherwise negative. All friction is assumed to be negligible.

* C. Fox, *Proc. Cambridge Phil. Soc.*, Vol. 45, Pt. 2 (1949), 311–315.

To A' is attached a horizontal rod or wire $A'B'$ and B' is then rotated about the vertical through P with angular velocity ω. A tension (or thrust) is thus generated in $A'B'$ which causes PA' to rotate about the vertical, also with angular velocity ω.

Our solution of the problem of closed navigation depends in part upon the measurement of this tension. We shall prove that the tension when suitably displayed upon a cathode ray tube screen gives us a circle in which (i) a specific diameter (or tangent) points in the same direction as a gyrocompass would at P, (ii) the radius is a function of λ, n and w. For slowly moving objects, such as ships at sea, the radius is frequently proportional to $\cos \lambda$ (see §7). The most convenient method of measuring this tension is probably by means of a resistance strain gage bonded to $A'B'$.

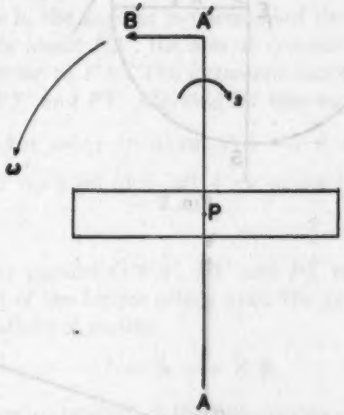

FIG. 1

The moment of the tension about PV, the upward drawn vertical through P, will be denoted by L. Its dynamical theory is discussed in the next section, §3.

To determine λ, ϕ, n and w we must have four measurements or four equations or the equivalent. The two equations given in §1 are already known so that two measurements suffice to solve the problem of closed navigation. One of these is given by the torque L and another by a second torque M generated by a gyroscope whose mounting and dynamics will be discussed in §5.

3. Dynamical Theory of Torque L. Figure 2 shows the body P situated on the earth's surface at the parallel of latitude λ. N and S are the north and south pole respectively and E is the earth's centre.

Relative to the earth P has (i) a westerly velocity w and consequently an angular velocity $-w/(a \cos \lambda)$ about SN and (ii) a northerly velocity n and consequently an angular velocity n/a about that earth's radius which is parallel to the direction due west at P. If we add to these the earth's angular velocity

about SN, denoted by Ω, we obtain the angular velocity of the body P in space. The components of this angular velocity in space about a line parallel to SN is then Ω', where

$$\Omega' = \Omega - (w/a \cos \lambda), \tag{1}$$

and about due west at P is n/a.

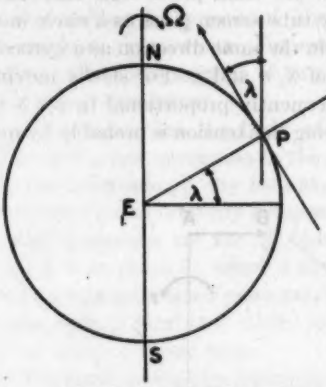

FIG. 2

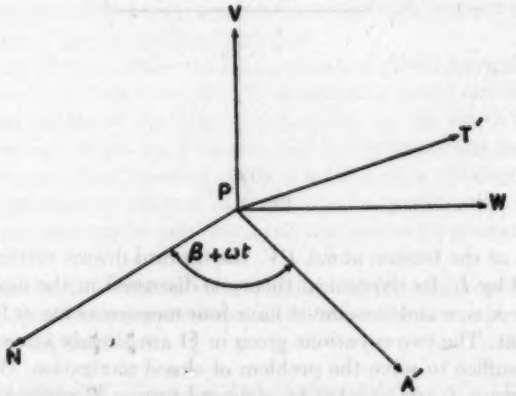

FIG. 3

Figure 3 shows a right handed system of axes, PN due north, PW due west and PV vertically upwards. The components of Ω' and n/a resolved along these axes are

$$(\Omega' \cos \lambda, n/a, \Omega' \sin \lambda), \tag{2}$$

where the first component is along PN, the second along PW and the third along PV.

Now the gyroscopic spin axis APA' is rotated about PV with angular velocity ω. Suppose that at time t PA' makes with PN (figure 3) an angle $\beta + \omega t$, where

β is unknown since the direction of PN is not known. Let PT' be that horizontal line which is perpendicular to PA' and which coincides with PW when PA' coincides with PN. Then PA', PT' and PV are principal axes of inertia for the gyroscope. Resolving (2) along the directions PA', PT' and PV and allowing for the spin ω about PV we see that $\boldsymbol{\omega}$, the spin of these axes in space, is given by

$$\begin{aligned}\boldsymbol{\omega} = [&\{\Omega' \cos\lambda \cos(\beta + \omega t) + (n/a) \sin(\beta + \omega t)\},\\ &\{-\Omega' \cos\lambda \sin(\beta + \omega t) + (n/a) \cos(\beta + \omega t)\},\\ &\{\omega + \Omega' \sin\lambda\}].\end{aligned} \tag{3}$$

The terms inside the first pair of brackets $\{\}$ give the component of $\boldsymbol{\omega}$ along PA', those inside the second pair the component along PT' and those inside the third pair the component along PV.

We now determine $\mathbf{h}$, the angular momentum of the gyroscope. Let I denote the moment of inertia about PA', the axis of symmetry, and C about any line through P perpendicular to PA'. The gyroscope has spin s about PA' relative to the system PA', PT' and PV. Allowing for this we find that

$$\begin{aligned}\mathbf{h} = [&I\{\Omega' \cos\lambda \cos(\beta + \omega t) + (n/a) \sin(\beta + \omega t) + s\},\\ &C\{-\Omega' \cos\lambda \sin(\beta + \omega t) + (n/a) \cos(\beta + \omega t)\},\\ &C\{\omega + \Omega' \sin\lambda\}].\end{aligned} \tag{4}$$

These components are parallel to PA', PT' and PV respectively.

If $\mathbf{L}$ is the moment of the torque acting upon the gyroscope we have, by the usual dynamical equations of motion

$$L = \dot{\mathbf{h}} + \boldsymbol{\omega} \times \mathbf{h}, \tag{5}$$

where $\boldsymbol{\omega} \times \mathbf{h}$ is the vector product of the two vectors $\boldsymbol{\omega}$ and $\mathbf{h}$. In forming these equations we note that Ω is small, since it is of the order $2\pi/(24 \times 60 \times 60)$ radians per second. Now Ω' is of the same order of smallness as Ω and both $\dot{\lambda} = n/a$ and w are much smaller. Hence squares and products of these quantities are completely negligible, a fact we make use of in writing down equation (6) below. The torque L, defined in §2, is the component of $\mathbf{L}$ parallel to PV. From (3), (4) and (5) we then have

$$L = -Is\{-\Omega' \cos\lambda \sin(\beta + \omega t) + (n/a) \cos(\beta + \omega t)\}. \tag{6}$$

4. The Determination of Gyronorth From (6). Suppose that the rotating gyroscope is connected to a cathode ray tube whose screen is maintained in a horizontal plane and whose display is given in the form of polar coordinates. The connections are such that the radius vector on the screen is (i) parallel to PA', the spin axis of the rotating gyroscope and (ii) proportional to L, the moment about PV of the tension in $A'B'$, defined in §2. We shall take the constant of proportionality to be unity. If the polar coordinates of the display point on the screen are (r, θ) then $r = L$ and $\theta = \beta + \omega t$. The last quantity is

the angle between the radius vector and due north at time t. From (6) the equation of the curve swept out on the screen is

$$r = Is\{\Omega' \cos\lambda \sin\theta - (n/a)\cos\theta\}. \tag{7}$$

This is evidently the polar equation of a circle which passes through the origin. The tangent to the circle at the origin makes with due north the angle given by

$$\tan\theta = n/a\Omega' \cos\lambda, \tag{8}$$

since $r = 0$ for this value of θ. Denoting the value of θ which satisfies (8) by α, we have

$$\tan\alpha = n/a\Omega' \cos\lambda. \tag{9}$$

The angle α is west of north if n is positive, i.e. if P is moving northwards and east of north if n is negative, i.e. P moving southwards. (9) is the well known equation for the speed and course (or meridian) error of a gyrocompass. Thus the tangent to circle (7) at the origin points in the same direction as the axis of a gyrocompass carried by the body P. When $n = 0$, i.e. when P is stationary or moves on a westerly or easterly course, the tangent points due north.

As in my previous paper I shall call this direction, which makes an angle α to the west of north, *gyronorth*. It plays as fundamental a part here as it does in gyrocompass theory.

In practice it may be easier to use the diameter of (7) which passes through the origin to give direction rather than the tangent to the circle at the origin. This diameter lies along a direction which makes an angle α to the south of due west, a direction I shall refer to as *gyrowest*. If P is at rest or moves on a westerly (or easterly) course then gyrowest coincides with due west. Gyrowest can be displayed on the cathode ray screen by making the electron beam trigger a time delay circuit as it passes through the origin. If the time delay is π/ω then, after leaving the origin, a point is arrived at which is half way round the circle, i.e. at the other end of the diameter through the origin. The circuit now triggered into action can be made to display the diameter required.

Since gyronorth is now known we may use it as a reference line. If the time is measured from the instant when the spin axis PA' coincides with gyronorth then at time t, PA' makes an angle $\beta + \omega t$ with true north and an angle ωt with gyronorth. Hence gyronorth will become our reference line if we take $\beta = \alpha$ in (6). We then have

$$L = Is\{\Omega' \cos\lambda \sin\alpha - (n/a)\cos\alpha\}\cos\omega t + Is\{\Omega' \cos\lambda \cos\alpha + (n/a)\sin\alpha\}\sin\omega t = Is\{\Omega' \cos\lambda \cos\alpha + \Omega' \cos\lambda \tan\alpha \sin\alpha\}\sin\omega t$$

on using (9)

$$= Is\Omega' \cos\lambda \sec\alpha \sin\omega t. \tag{10}$$

The equation of the corresponding display on the horizontal screen of the cathode ray tube is then

$$r = Is\Omega' \cos\lambda \sec\alpha \sin\theta, \tag{11}$$

where θ is now the angle measured from gyronorth, [instead of from true north as in (7)].

Although gyronorth is known yet true north is still unknown, as in the case of the gyrocompass. Hence α counts as an unknown quantity. But the introduction of α is compensated by the introduction of the additional equation (9). For the problem of closed navigation, as now envisaged, requires the determination of four unknowns λ, n, w and α, among which exist two relations $\dot{\lambda} = n/a$ and (9). Thus two measurements still suffice for its solution, e.g. the torques L, §3, and M, §5.

To obtain an idea of the magnitude of L consider the case of a gyroscope whose wheel has a mass of 50 lbs. and diameter 1 ft. Taking the spin $s = 10{,}000$ revolutions per minute and $\Omega = \Omega' = 2\pi/(24 \times 60 \times 60)$ radians per second we find that $Is\Omega'$ is a torque of order .4 (poundals and feet). Thus the tension in $A'B'$ which generates L should be capable of measurement by strain gages, or by many other means, to a high degree of accuracy.

Many methods of measurement enable us to differentiate L with respect to time. For example if L appears as a voltage it can be passed through a differentiating circuit. From (10) we then have

$$dL/dt = \omega Is\Omega' \cos \lambda \sec \alpha \cos \omega t. \tag{12}$$

If this is displayed upon the horizontal screen of a cathode ray tube with dL/dt as the radius vector and ωt, measured from gyronorth, as the vectorial angle we obtain the circle

$$r = \omega Is\Omega' \cos \lambda \sec \alpha \cos \theta. \tag{13}$$

The diameter of this circle which passes through the origin is parallel to gyronorth. Although (13) can be used instead of circle (7) and may offer pratical advantages in some cases it must be noted that the gimbal bearings are required to resist a torque of magnitude $Is\omega$, hence ω cannot be made too large. This fact follows easily on calculating the torque about PT' from (3), (4) and (5).

5. A Second Gyroscopic System. To solve the problem of closed navigation a second measurement is required in addition to L. This is now dealt with.

We can now make use of the fact that the direction gyronorth is known. In figure 4 PN is due north, PW due west and PV vertically upwards. Of these only the direction PV is known. $PG{\cdot}N$ is gyronorth, i.e. at α to the west of true north and $PG{\cdot}W$ is gyrowest, at α to the south of west. Both these directions are known and α is given by (9). Now the spin in space of the system PN, PW, PV is given by (2). So to find the spin of the system $PG{\cdot}N$, $PG{\cdot}W$ and PV we must resolve (2) along these directions and allow for the extra spin $\dot{\alpha}$ about PV. We then obtain

$$\begin{aligned}&[\{\Omega' \cos \lambda \cos \alpha + (n/a) \sin \alpha\},\\ &\qquad \{-\Omega' \cos \lambda \sin \alpha + (n/a) \cos \alpha\}, \{\dot{\alpha} + \Omega' \sin \lambda\}],\end{aligned} \tag{14}$$

where the first component is parallel to $PG{\cdot}N$, the second to $PG{\cdot}W$ and the

third to PV. Confining ourselves to the case when P pursues a steady course, so that n is constant, on differentiating (9) with respect to time we see that $\dot{\alpha}$ is proportional to $n\dot{\lambda}/a$, i.e. to $\dot{\lambda}^2$. Hence, as shown in §3, $\dot{\alpha}$ is negligible in comparison with Ω'. On ignoring $\dot{\alpha}$ and using (9) the system (14) reduces to

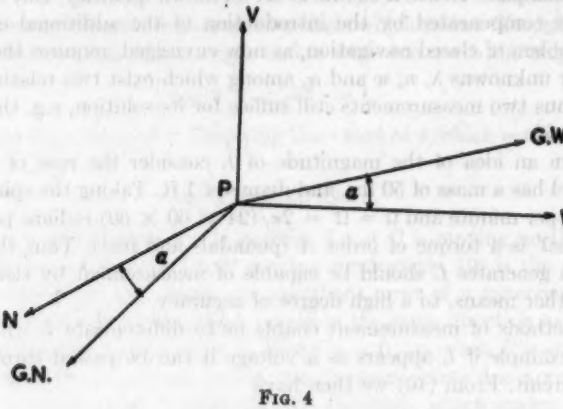

FIG. 4

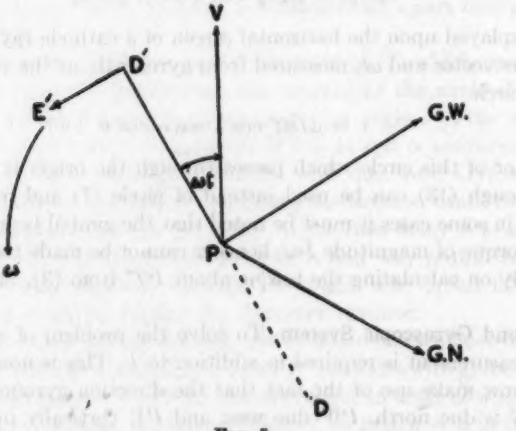

FIG. 5

$$(\Omega' \cos\lambda \sec\alpha,\ 0,\ \Omega' \sin\lambda). \tag{15}$$

A gyroscope is now mounted so that its spin axis DD' can swing freely in a vertical plane perpendicular to $PG{\cdot}N$, i.e. perpendicular to gyronorth. Figure 5 shows the three axes $PG{\cdot}N$, $PG{\cdot}W$, PV and also the spin axis DPD'. To D' is attached a rod or wire $D'E'$ which carries a strain gage or other suitable means of measuring tension (or thrust) in $D'E'$. The point E' is then revolved about the axis $PG{\cdot}N$ with angular velocity ω. A tension (or thrust) is thus generated in

$D'E'$ which causes PD to rotate about $PG{\cdot}N$ also with angular velocity ω. The measurement of this tension enables us to complete the solution of the problem of closed navigation.

The moment of the tension about $PG{\cdot}N$ will be denoted by M. In the dynamical theory of M, given below, we assume that the gyroscope with spin axis DPD' has the same moments of inertia I and C and the same spin s and the same ω as for the gyroscope with spin axis APA'. This assumption, however, although convenient is not essential to our theory.

Since the vertical is known we can measure time from the instant when PD' coincides with PV so that the angle VPD' is then equal to ωt. We shall consider ω to be positive if the rotation when viewed from P towards $G{\cdot}N$ (figure 5) is clockwise, otherwise it will be negative.

The principal axes of inertia of the gyroscope are $PG{\cdot}N$, PD' and a line PX perpendicular to both of these directions. From (15) ω', the spin in space of these axes is given by

$$\omega' = [\{\omega + \Omega' \cos \lambda \sec \alpha\}, \Omega' \sin \lambda \cos \omega t, -\Omega' \sin \lambda \sin \omega t]. \quad (16)$$

This has been obtained by resolving the angular velocities of (15) along the directions $PG{\cdot}N$, PD and PX and allowing for the extra spin ω about $PG{\cdot}N$.

On allowing for the spin s about PD' the angular momentum $\mathbf{h}'$ of the gyroscopic system is easily found. Its components along $PG{\cdot}N$, PD and PX are given by

$$\mathbf{h}' = \{C(\omega + \Omega' \cos \lambda \sec \alpha), I(s + \Omega' \sin \lambda \cos \omega t), -C\Omega' \sin \lambda \cos \omega t\}. \quad (17)$$

If $\mathbf{M}$ is the moment of the couple acting on this system then the usual dynamical equations of motion are

$$\mathbf{M} = \dot{\mathbf{h}}' + \omega' \times \mathbf{h}'.$$

In forming these equations, we shall, as in §3, ignore the very small second order terms Ω'^2, $\Omega'\dot{\lambda}$, $\Omega' n/a$ etc. M, the moment of the tension in $D'E'$ about $PG{\cdot}N$, is the component of $\mathbf{M}$ along $PG{\cdot}N$. Its equation is therefore

$$M = Is\Omega' \sin \lambda \sin \omega t. \quad (18)$$

6. The Problem of Closed Navigation. We have now sufficient data to solve the problem of closed navigation, in theory at least. Since gyronorth is known we have four unknowns to determine. They are α, the angle between gyronorth and true north; λ the latitude; n and w the northerly and westerly components of velocity, respectively. For their determination we have four equations. The first is the obvious equation

$$\dot{\lambda} = n/a \quad (19)$$

where a is the earth's radius. The second is equation (9) which we repeat for convenience,

$$\tan \alpha = n/a\Omega' \cos \lambda \quad (20)$$

The other two equations are given by (10) and (18). In practice L and M will either be measured by an electrical method, appearing as an A.C. current or voltage, or, if measured by any other means, can be transformed into an A.C. current or voltage. It is therefore possible to determine the amplitudes of the expressions on the right hand sides of (10) and (18). Denoting these amplitudes by L_1 and M_1 respectively, we have from (10)

$$L_1 = Is\Omega' \cos \lambda \sec \alpha \tag{21}$$

and from (18)

$$M_1 = Is\Omega' \sin \lambda \tag{22}$$

From (19), (20) and (21) we deduce that

$$L_1 = Is\dot{\lambda} \operatorname{cosec} \alpha \tag{23}$$

and from (19), (20) and (22) that

$$M_1 = Is\dot{\lambda} \cot \alpha \tan \lambda. \tag{24}$$

Hence, on eliminating α from (23) and (24) we have

$$Is\dot{\lambda} = (L_1^2 - M_1^2 \cot^2 \lambda)^{1/2}. \tag{25}$$

This is the same as equation (14) of my previous paper* on allowing for some difference in the terminology of the two papers.

The solution of the problem of closed navigation, as developed here, depends upon designing a computor which is capable of integrating (24). Since L_1 and M_1 vary as P moves over the earth's surface (25) is not a simple equation to integrate. For slowly moving bodies, such as ships at sea, these formulae simplify considerably and λ can be found without the need of integrating (25) as we shall show in the next section.

To continue with the solution of our problem after finding λ and $\dot{\lambda}$ from (25), we can, in theory, find n from (19) and then α from (23) and so determine true north. From (22) we then find Ω' and from (1) $w/(a \cos \lambda)$ and so w. By integating $w/(a \cos \lambda)$ with respect to time we can determine longitude except for an arbitrary constant, since $\dot{\phi} = w/(a \cos \lambda)$.

The conclusions arrived at here are in complete accord with those of my previous paper. But the methods proposed here for measuring L_1 and M_1 enable us to dispense with the use of a gyrocompass. In fact the gyroscope whose spin axis rotates in a horizontal plane includes among its other functions that of the gyrocompass.

7. Applications to Navigational Problems. For vessels travelling at not more than 20 knots or so in latitudes where $-60° < \lambda < 60°$ the value of α is very small. The largest value occurs when the vessel is travelling due north or due south and when $\lambda = \pm 60°$. Taking $\Omega' = \Omega = 2\pi/(24 \times 60 \times 60)$ radians per second and a (the earth's radius) = 3960 statute miles we find from (20) that

* loc. cit., p. 314.

$\alpha = 2°33'$ and hence sec $\alpha = 1.0010$. The error in taking sec $\alpha = 1$ is not more than .1% and in most cases is much less. It then follows from (21) and (22) that

$$\tan \lambda = M_1/L_1 \tag{26}$$

is accurate to two or three minutes of arc. If the vessel is at rest or travelling due east or west then (26) is completely accurate. For vessels moving on an easterly or westerly course the error in λ on using (26) will be less than a fraction of a minute.

From (10) and (18) it follows that under these circumstances we also have

$$\tan \lambda = M/L = (dM/dt)/(dL/dt) \tag{27}$$

Having found λ we can then find Ω' from (22) and so w from (1). But for ships at sea $w/(a \cos \lambda)$ may be of order $\Omega/50$ and so might be too small for practical determination. But the conclusions above also apply to aircraft flying on an easterly or westerly course provided that their northerly (or southerly) velocity component does not exceed 20 knots. For the value of λ deduced from (26) would still remain correct to two or three minutes. But for aircraft flight $w/(a \cos \lambda)$ might bear a high ratio to the value of Ω and it would then become possible to find w by the means proposed above.

In the case of vessels at rest or travelling due north or south we have $w = 0$ and so from (1) $\Omega' = \Omega$. Equation (22) then enables us to find λ accurately whatever the speed of the vessel. Evidently for a vessel with a sufficiently small easterly or westerly velocity component (22) in the approximate form $M_1 = Is\Omega \sin \lambda$ still gives λ to a high degree of accuracy.

We have already noticed that for a ship travelling at a speed not in excess of 20 knots in latitudes between $\pm 60°$ we may write sec $\alpha = 1$. If in addition it is moving due north or due south we have $\Omega' = \Omega$, from (1) and so equation (21) becomes $L_1 = Is\Omega \cos \lambda$. We can then find λ accurately by using one gyroscopic system only, namely the system described in §2. In fact there is no need even to find L_1 for the equation of circle (11) becomes

$$r = Is\Omega \cos \lambda \sin \theta. \tag{28}$$

Hence the circle displayed on the cathode ray screen has a diameter proportional to cos λ and this enables us to determine λ.

These results assume that the ship travels due north or due south and if this is not the case they are no longer precise. But if the ship's easterly or westerly velocity component is sufficiently small the method outlined above still gives λ to a high degree of approximation.

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ON THE DEFINITION OF INFORMATION

By Edgar Reich

Summary. A definition of information is proposed by postulating an invariance under certain types of transformations. For a restricted class of admissible definitions these postulates imply that Shannon's formulation is the only possible one.

During the past several years a statistical theory of information has been developed by Shannon[1] and Wiener[2] in which the entropy function of thermodynamics plays a predominant part. "Information," as used in the theory, is a scalar associated with the message whose information-content is under consideration. The value of the scalar is made to depend both on the message and the a-priori knowledge of the event with which the message deals. It is useful to think of the message as something that transmits data regarding the occurrence of a physical event, the transmission taking place in the presence of "noise." Thus, for example, we can consider a meter as giving "information" about the physical quantity it measures. Here the physical quantity is the "event," the inaccuracies correspond to "noise," and reading, together with a statement of the inaccuracies, constitutes the "message." There are obviously many forms a message may assume; for instance, in particular cases it may simply consist of a physiological reaction obtained through the human senses.

In a given physical situation there may not be a perfectly unique way of determining what constitutes the event, and what the message. To be more exact, therefore, one should not use merely the word, "information," but instead the phrase, "information about event E of a class of events C conveyed by message M relative to a-priori knowledge K."

Information Rate of a Random Process.

It is questionable whether a theory of information can be obtained unless both the events and noise are assumed to be generated by a random process.[3] The interpretation for the case of comunication in the presence of noise is:

source symbol $\leftrightarrow$ event,

received symbol (determined by source symbol and noise symbol) $\leftrightarrow$ message,

class of events $\leftrightarrow$ set of possible source symbols,

a-priori knowledge $\leftrightarrow$ a-priori probability that a certain symbol will be sent.

For the sake of simplicity it will henceforth be assumed that the source symbols are real numbers, and that successive symbols are independent. It is also assumed that noise affects successive symbols independently. In this case the received symbols will be real numbers, successively independent, and the following description is possible:

[1] C. E. Shannon, "A Mathematical Theory of Communication," Bell System Technical Journal, 1948, pp. 379–423, and 1948, pp. 623–656.

[2] N. Wiener, "Cybernetics," New York, 1948, pp. 74–81.

[3] Cf. C. E. Shannon, "Communication Theory of Secrecy Systems," Bell System Technical Journal, 1949, pp. 663, 4.

$p(x, y)$ = probability density that x is transmitted *and* y received,

$p_0(x) = \int p(x, y)\, dy$ = a-priori probability density that x was transmitted,

$q(y) = \int p(x, y)\, dx$ = a-priori probability density that y will be received (based on ignorance of x),

$p_m(x) = p(x, m)/q(m)$ = probability density that x was transmitted a-posteriori to receipt of $y = m$.

Let $I(m)$ denote the information obtained as the result of receiving $y = m$. The following postulate immediately suggests itself:

POSTULATE I: $I(m)$ is a scalar functional of $p_0(x)$ and $p_m(x)$

$$I(m) = \Phi[p_0(x), p_m(x)]$$

with the property that

$$\Phi[p_0(x), p_0(x)] = 0.$$

The second part of the postulate implies that no information was gained if the a-posteriori probability as to what *was* transmitted is the same as the a-priori one, as to what *will* be transmitted.

As the entire process we are considering is a statistical one it is to be expected that statistical functions of I will play a more important part than I itself. We define the "information rate" R as the *average* amount of information received *per transmission*, i.e. as the expected value of I:

$$R = E[I(m)] = \int q(m) I(m)\, dm.$$

R should be invariant under any transformation that merely amounts to a one-to-one relabeling of the message symbols without changing the fundamental physical process, otherwise the information obtainable from a message could be changed by restating the message in a logically equivalent way. For instance, suppose the received message is read from a meter calibrated according to y^3 instead of y. If the distributions p_0 and p_m are recalculated on the basis of x^3 instead of x the resulting value of R should be the same. Now a relabeling of the variable x to $f(x)$, where $f(x)$ is differentiable, transforms a distribution $p(x)$ into the distribution $p(x)/f'(x)$ where $x = g(z)$ is the function inverse to $z = f(x)$, which is assumed monotonically increasing. Therefore we have the following postulate:

POSTULATE II: The transformation

$p(x) \rightarrow p(g(x))/f'(g(x))$ where g is the inverse of f, and p generally represents all the probability distributions entering into the definition of R, *leaves R invariant.*

A Uniqueness Theorem for a Restricted Class of Definitions.

It is possible to provide a mechanism for the process of information receipt by considering the recipient to have an a-priori uncertainty $U_0 = U(p_0(x))$ as

to what *will* be transmitted, and an a-posteriori uncertainty $U_m = U(p_m(x))$ as to what *was* transmitted. A restricted type of definition of I is therefore one of the form $I = U_0 - U_m$. It is to be noted that U is some scalar quantity associated with a distribution function, giving a measure of the uncertainty accompanying this distribution. Presumably the narrower the distribution the lower the value of U. There are of course many ways in which a scalar can be associated with a distribution, but this paper will henceforth concern itself only with functionals of the type

$$U = \int F(p, x)\, dx$$

where $F = F(u, v)$ is some real function of two real variables. This is a common method for assigning scalars to distribution functions; for instance the k'th moment of p is obtained by setting $F(u, v) = uv^k$.

It will be assumed that F has continuous partial derivatives through the second order, and in fact we will from now on assume all sorts of "good behavior," including interchangeability of order of integration, etc. With these limitations in mind the following uniqueness theorem will be derived:

THEOREM: If I is of the form $I = U_0 - U_m$, and U_i is of the form $U_i = \int F(p_i(x), x)\, dx$ then in order that postulates I and II are satisfied it is necessary that

$$R = \text{constant} \cdot \iint p(x, y) \log \frac{p(x, y)}{p_0(x)q(y)}\, dx\, dy.^4$$

PROOF: Since the definition of U automatically implies that postulate I is satisfied it is only necessary to subject R to the conditions of postulate II. We have

$$R = \int F(p_0, x)\, dx - \iint q(m)F(p_m, x)\, dx\, dm. \tag{1}$$

The invariance condition implies that

$$\int F\left[\frac{p_0(g(x))}{f'(g(x))}, x\right] dx - \iint \frac{q(g(m))}{f'(g(m))} F\left[\frac{p_m(g(x))}{f'(g(x))}, x\right] dx\, dm$$
$$= \int F\left[\frac{p_0(x)}{f'(x)}, f(x)\right] f'(x)\, dx - \iint q(m)F\left[\frac{p_m(x)}{f'(x)}, f(x)\right] f'(x)\, dx\, dm \tag{2}$$

is independent of the choice of f. Let

$$F(u, v) = uG(u, v). \tag{3}$$

Then (2) becomes

$$\int p_0 G(p_0/f', f)\, dx - \iint q p_m G(p_m/f', f)\, dx\, dm. \tag{4}$$

[4] This is the formula for information rate proposed by Shannon on the basis of considerations other than those employed here.

Subject f to the variation

$$\Delta f(x) = \epsilon w(x). \tag{5}$$

Since (4) is independent of f the corresponding variation of (4) must vanish:

$$-\int p_0^2 G_u^0 w'/f'^2\,dx - \iint q p_m^2 G_u^m w'/f'^2\,dx\,dm \tag{6.1}$$

$$+\int p_0 G_v^0 w\,dx + \iint q p_m G_v^m w\,dx\,dm = 0 \tag{6.2}$$

Since w' can be very large compared to w lines (6.1) and (6.2) must vanish separately. The vanishing of (6.1) implies in turn that

$$p_0^2 G_u(p_0/f', f) + \int q p_m^2 G_u(p_m/f', f)\,dm = 0 \tag{7}$$

Setting

$$G_u(u, v) = r(u, v)/u^2 \tag{8}$$

we obtain

$$r(p_0/f', f) + \int q r(p_m/f', f)\,dm = 0 \tag{9}$$

Note that a variation $\Delta p(x,y) = \epsilon h(x)k(y)$, where $\int h\,dx = \int k\,dx = 0$, is an admissible variation of $p(x, y)$ providing h and k are appropriately bounded. Such a variation produces the following variations in the associated distributions:

$$\Delta p_0(x) = 0, \qquad \Delta q(m) = 0, \qquad \Delta p_m(x) = \epsilon h(x)k(m)/q(m). \tag{10}$$

Subjecting the respective quantities of (9) to the variations prescribed by (10) yields

$$\int r_u(p_m/f', f)k(m)\,dm = 0 \tag{11}$$

Due to the arbitrariness of $k(m)$ (11) implies that

$$r_u(p_m/f', f) = \text{function independent of } m. \tag{12}$$

Subjecting (12) to another variation of the type (10) we obtain (12) + Δ(12) = independent of m;

therefore

$$f'\Delta(12) = \epsilon r_{uu}(p_m/f', f)h(x)k(m)/q(m) = \text{indep. of } m. \tag{13}$$

From the arbitrariness of $k(m)$ it clearly follows that

$$r_{uu}(u, v) = 0 \tag{14}$$

Combining (14) and (8) gives

$$G(u, v) = a(v)\ln u + b(v)/u + c(v) \tag{15}$$

for some functions $a(v)$, $b(v)$, $c(v)$.
Returning now to (6.2) we see that it implies

$$p_0 G_*(p_0/f', f) + \int q p_m G_*(p_m/f', f)\, dm = 0 \tag{16}$$

Let

$$G_*(u, v) = s(u, v)/u. \tag{17}$$

Then (16) becomes

$$s(p_0/f', f) + \int q s(p_m/f', f)\, dm = 0. \tag{18}$$

As (18) is of exactly the same form as (9) it similarly implies that

$$s_{uu}(u, v) = 0. \tag{19}$$

Combining (19) and (17) yields

$$G_*(u, v) = A(v) + B(v)/u. \tag{20}$$

But (15) implies

$$G_*(u, v) = a'(v)\ln u + b'(v)/u + c'(v). \tag{21}$$

Combining (20) and (21):

$$a'(v) = 0. \tag{22}$$

Thus

$$G(u, v) = \text{const}\cdot\, \ln u + b(v)/u + c(v) \tag{23}$$

and

$$F(u, v) = \text{const}\cdot\, u\ln u + uc(v) + b(v). \tag{24}$$

Substituting (24) in (1):

$$R/\text{const} = \int p_0 \ln p_0\, dx - \iint q p_m \ln p_m\, dx\, dm + \int \left(p_0 - \int q p_m\, dm\right) c(x)\, dx \tag{25}$$

But

$$\int q(m) p_m(x)\, dm = \int q(m) p(x, m)/q(m)\, dm = p_0(x).$$

Therefore the last integral of (25) vanishes, making it

$$R/\text{const} = \iint p(x, y) ln p_0 \, dx \, dy - \iint p(x, m) ln[p(x, m)/q(m)] \, dx \, dm$$

(26)

$$= - \iint p(x, y) \, ln[p(x, y)/p_0(x)q(y)] \, dx \, dy,$$

as was to be proved.

In addition to the preceding study a class of definitions for information rate of the type

$$R = \iint F(p_0(x), p_m(y)) \, dx \, dy,$$

where F is again some real function of two real variables, was considered. A heuristic derivation indicates that again the result (26) is implied. Because of its inelegance and length this work is not presented here.

These various specific results lead one to suspect that a corresponding general result may exist, but it appears that a different approach would be required to obtain the latter.

It should be stressed that the postulational approach to the definition of information given in this paper is by no means the only such postulational method possible. One of the customary alternate ideas is to postulate an additivity for the total information output of independent information sources. Together with other assumptions of a simple kind additivity postulates of this type imply a unique formulation, at least for the case of discrete probabilities.

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[4] Cf. "A Mathematical Theory of Communication," loc. cit, Appendix 2.

TABLE OF THE INTEGRAL, $\int_0^x e^{-u^3}\,du$

By Milton Abramowitz

The present article is devoted to the tabulation of the integral

$$I(x) = \int_0^x e^{-u^3}\,du.$$

This function occurs in the solution of the equation for heat conduction in the case of a fluid flowing past a flat plate.[1] The integral has been tabulated at intervals of .01 to eighth decimal places with second central differences. Since

$$\int_0^\infty e^{-u^3}\,du = \frac{\Gamma(1/3)}{3} = .89297\ 95116 \text{ (correct to 10 decimal places)}$$

$I(x)$ has been tabulated to the point $x = 2.50$ where it differs from $\Gamma(1/3)/3$ by one unit in the eighth decimal place.

The tabular values were computed to ten decimal places from the power series

$$I(x) = x - \frac{1}{4}x^4 + \frac{1}{7}\frac{x^7}{2!} - \frac{1}{10}\frac{x^{10}}{3!} + \cdots$$

in the range $x = 0$ to $x = 1.5$. An independent check was made for $x = 1.5$ with the aid of the expansion

$$I(x) = e^{-x^3}\left\{x + \frac{3x^4}{4} + \frac{3^2x^7}{4.7} + \frac{3^3x^{10}}{4.7.10} + \cdots\right\}$$

For $x > 1.5$, $I(x)$ was obtained by numerical integration of the function e^{-u^3}. Independent checks were made with the asymptotic expansion

$$I(x) = \frac{\Gamma(1/3)}{3} - \frac{e^{-x^3}}{3x^2}\left[1 - \frac{2}{3x^3} + \frac{2.5}{3^2x^6} - \frac{2.5.8}{3^3x^9} + \cdots + \frac{(-1)^n 2.5.8\cdots(3n-1)}{(3x^3)^n}\right] + R_n$$

where

$$R_n = \frac{(-1)^n(3x^2e^{x^3})[2.5.8\cdots(3n-1)(3n+2)]}{3^{n+1}}\int_x^\infty \frac{e^{-u^3}\,du}{u^{3n+2}}$$

The ten place values were differenced and ultimately rounded to 8 decimal places. The tabulated second differences are those obtained by rounding the differences of the ten-place values. Linear interpolation will generally be good to four decimal places.

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[1] L. M. K. Boelter and others, *Heat transfer notes*, (University of California Press, Berkeley, Calif., 1948).

TABLE 1

x	$\int_0^x e^{-u^2}du$	$-b_1$	x	$\int_0^x e^{-u^2}du$	$-b_1$	x	$\int_0^x e^{-u^2}du$	$-b_1$	x	$\int_0^x e^{-u^2}du$	$-b_1$	x	$\int_0^x e^{-u^2}du$	$\frac{b_1}{1}$
0.00	0.0000 0000	0	0.50	0.4849 1714	6619	1.00	0.8075 1118	11036	1.50	0.8886 1722	2310	2.00	0.8929 5351	40
.01	.0100 0000	4	.51	.4937 0866	6634	.01	.8111 3498	10922	.51	.8889 4806	2187	.01	5667	36
.02	.0199 9996	12	.52	.5024 3184	7048	.02	.8146 4957	10800	.52	.8892 5703	2069	.02	5947	32
.03	.0299 9980	28	.53	.5110 8454	7261	.03	.8180 5615	10671	.53	.8895 4531	1955	.03	6195	29
.04	.0399 9936	48	.54	.5196 6463	7473	.04	.8213 5602	10535	.54	.8898 1404	1845	.04	6414	26
0.05	0.0499 9844	76	0.55	0.5281 6999	7684	1.05	0.8245 5055	10393	1.55	0.8900 6431	1740	2.05	0.8929 6607	23
.06	.0599 9676	108	.56	.5365 9851	7893	.06	.8276 4114	10244	.56	02 9718	1639	.06	6778	20
.07	.0699 9400	147	.57	.5449 4810	8099	.07	.8306 2930	10089	.57	05 1366	1543	.07	6928	18
.08	.0799 8976	192	.58	.5532 1670	8303	.08	.8335 1657	9928	.58	07 1471	1451	.08	7060	16
.09	.0899 8360	243	.59	.5614 0237	8504	.09	.8363 0456	9762	.59	09 0125	1362	.09	7175	14
0.10	0.0999 7501	300	0.60	0.5695 0281	8702	1.10	0.8389 9493	9591	1.60	0.8910 7417	1278	2.10	0.8929 7277	13
.11	.1099 6341	363	.61	.5775 1633	8896	.11	.8415 8940	9414	.61	12 3431	1198	.11	7366	11
.12	.1199 4819	432	.62	.5854 4089	9086	.12	.8440 8972	9234	.62	13 8247	1122	.12	7444	10
.13	.1299 2884	506	.63	.5932 7458	9272	.13	.8464 9770	9049	.63	15 1941	1049	.13	7512	9
.14	.1399 0404	587	.64	.6010 1556	9454	.14	.8488 1519	8861	.64	16 4586	980	.14	7571	8
0.15	0.1498 7386	673	0.65	0.6086 6199	9631	1.15	0.8510 4406	8670	1.65	0.8917 6250	915	2.15	0.8929 7623	7
.16	.1598 3635	765	.66	.6162 1212	9802	.16	.8531 8624	8475	.66	18 7001	853	.16	7668	6
.17	.1697 9149	863	.67	.6236 6432	9969	.17	.8552 4367	8278	.67	19 6898	794	.17	7707	5
.18	.1797 3800	967	.68	.6310 1664	10129	.18	.8572 1833	8078	.68	20 6001	739	.18	7741	5
.19	.1896 7484	1076	.69	.6382 6776	10283	.19	.8591 1220	7877	.69	21 4364	687	.19	7771	4
0.20	0.1996 0091	1191	0.70	0.6454 1606	10431	1.20	0.8609 2730	7674	1.70	0.8922 2042	638	2.20	0.8929 7797	3
.21	.2095 1508	1311	.71	.6524 6005	10572	.21	.8626 6566	7470	.71	22 9081	591	.21	7819	3
.22	.2194 1614	1437	.72	.6593 9831	10707	.22	.8643 2933	7265	.72	23 5530	548	.22	7838	3
.23	.2293 0282	1568	.73	.6662 2930	10834	.23	.8659 2035	7059	.73	24 1430	507	.23	7854	2
.24	.2391 7383	1705	.74	.6729 5235	10954	.24	.8674 4077	6854	.74	24 6824	468	.24	7868	2
0.25	0.2490 2778	1846	0.75	0.6795 6566	11066	1.25	0.8688 9265	6648	1.75	0.8925 1750	432	2.25	0.8929 7881	2
.26	.2588 6337	1993	.76	.6860 6831	11171	.26	.8702 7806	6443	.76	25 6244	399	.26	7891	1
.27	.2686 7884	2145	.77	.6924 5925	11267	.27	.8715 9902	6239	.77	26 0338	367	.27	7900	1
.28	.2784 7295	2301	.78	.6987 3732	11355	.28	.8728 5760	6036	.78	26 4066	338	.28	7908	1
.29	.2882 4405	2463	.79	.7049 0225	11435	.29	.8740 5581	5835	.79	26 7455	311	.29	7914	1
0.30	0.2979 9052	2628	0.80	0.7109 5362	11506	1.30	0.8751 9568	5635	1.80	0.8927 0534	285	2.30	0.8929 7920	1
.31	.3077 1071	2799	.81	.7168 8794	11568	.31	.8762 7920	5437	.81	27 3328	261	.31	7925	1
.32	.3174 0292	2973	.82	.7227 0755	11622	.32	.8773 0835	5241	.82	27 5860	240	.32	7929	1
.33	.3270 6539	3152	.83	.7284 1100	11666	.33	.8782 8509	5048	.83	27 8153	219	.33	7932	1
.34	.3366 9634	3335	.84	.7339 9776	11702	.34	.8792 1135	4857	.84	28 0227	200	.34	7935	0
0.35	0.3462 9394	3521	0.85	0.7394 6750	11728	1.35	0.8800 8903	4670	1.85	0.8928 2100	183	2.35	0.8929 7938	
.36	.3558 5633	3711	.86	.7448 1997	11745	.36	.8809 2003	4485	.86	28 3790	167	.36	7940	
.37	.3653 8161	3904	.87	.7500 5498	11753	.37	.8817 0617	4304	.87	28 5314	152	.37	7942	
.38	.3748 6785	4101	.88	.7551 7247	11752	.38	.8824 4927	4126	.88	28 6686	138	.38	7943	
.39	.3843 1308	4300	.89	.7601 7244	11741	.39	.8831 5110	3952	.89	28 7921	125	.39	7945	
0.40	0.3937 1539	4503	0.90	0.7650 5500	11721	1.40	0.8838 1342	3782	1.90	0.8928 9029	114	2.40	0.8929 7946	
.41	.4030 7250	4707	.91	.7698 2034	11692	.41	.8844 3792	3616	.91	29 0024	103	.41	7947	
.42	.4123 8262	4914	.92	.7744 6877	11654	.42	.8850 2626	3453	.92	29 0916	93	.42	7947	
.43	.4216 4361	5123	.93	.7790 0064	11607	.43	.8855 8006	3295	.93	29 1714	84	.43	7948	
.44	.4308 5336	5334	.94	.7834 1648	11551	.44	.8861 0092	3141	.94	29 2428	76	.44	7948	
0.45	0.4400 0977	5546	0.95	0.7877 1674	11487	1.45	0.8865 9037	2992	1.95	0.8929 3066	69	2.45	0.8929 7949	
.46	.4491 1072	5759	.96	.7919 0216	11413	.46	.8870 4990	2846	.96	29 3635	62	.46	7949	
.47	.4581 5408	5974	.97	.7959 7346	11331	.47	.8874 8096	2706	.97	29 4142	56	.47	7950	
.48	.4671 3770	6188	.98	.7999 3143	11241	.48	.8878 8497	2569	.98	29 4594	50	.48	7950	
.49	.4760 5944	6404	.99	.8037 7703	11142	.49	.8882 6328	2437	.99	29 4995	45	.49	7950	
0.50	0.4849 1714	6619	1.00	0.8075 1118	11036	1.50	0.8886 1722	2310	2.00	0.8929 5351	40	2.50	0.8929 7950	

SOME REMARKS ON THE COUPLING OF TWO DUCTS

By Albert E. Heins

1. Introduction. A recent paper by Weitz and Keller (1) studies the effect of surface waves on two coupled channels of finite depth. Each surface on each of these respective channels has different physical properties and hence different boundary conditions are in effect for $x > 0$ and $x < 0$. To describe the geometry more adequately we have the infinite strip $-\infty < x < \infty$, $0 \leq y \leq a$. For $y = a$, $x > 0$ there is a given boundary condition, for $y = a$, $x < 0$ there is a second boundary condition and for all x and $y = 0$ there is a third. Problems of this nature have already been considered by Feshbach and Heins in 1947 (2), and by Heins in 1948 (3) and 1950 (4). Such boundary value problems are of the mixed type because two different boundary conditions are in force on the line $y = a$. The experience of Feshbach and Heins has indicated that the method of singular integral equations of the Wiener-Hopf type is an appropriate form in which to formulate these problems. Weitz and Keller do not get the Wiener-Hopf integral equation and remark that the methods of Feshbach and Heins are inapplicable. This is the case because an appropriate Green's function was not employed by these authors. If care is taken in selecting the Green's function we obtain an integral equation which is completely reminiscent of the investigations of Feshbach and Heins. We shall refer freely in this note to the work already available (2, 3, 4).

2. Statement of the Problem. The general mathematical problem of the coupling of two ducts is dependent on the partial differential equation

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 - k^2\phi = 0 \tag{2.1}$$

subject to the boundary conditions

$$\partial\phi/\partial y = 0 \qquad y = 0, \qquad -\infty < x < \infty \tag{2.2}$$

$$\partial\phi/\partial y = \sigma_1\phi \qquad y = a, \qquad x < 0 \tag{2.3}$$

$$\partial\phi/\partial y = \sigma_2\phi \qquad y = a, \qquad x > 0 \tag{2.4}$$

For k real we have the problem of surface waves while for k imaginary we deal with acoustic waves. Various cases of the parameters σ_1 and σ_2 have been treated. For σ_1 and σ_2 complex, and k imaginary we have the work of Feshbach and Heins (2). For $\sigma_1 > 0$, $\sigma_2 = 0$ and k real we have the "dock" problem (3).

In addition to the conditions (2.2), (2.3) and (2.4), conditions at infinity are required. We shall suppose that for $x \to -\infty$ (k real)

$$\phi(x, y) = [\alpha_1 \exp(ikx) + \beta_1 \exp(-ikx)] \cosh \rho_0 y/a \tag{2.5}$$

where

$$\kappa = \sqrt{(\rho_0^2/a^2) - k^2}$$

and ρ_0 is the real positive root of

$$\sigma_1 a \cosh \rho - \sinh \rho = 0 \tag{2.6}$$

It is assumed that σ_1 is sufficiently large and positive to insure that $\varkappa$ be real. (2.5) is the asymptotic form of $\phi(x, y)$ for $x \to -\infty$ under the assumption that there is propagation in the left duct. Here we see that in the assumption (2.5) we are making a specific statement regarding the form of $\phi(x, y)$ for $x \to -\infty$. While it might be more desirable from a mathematical standpoint to merely require an order condition [see (1), eq. (2.5) and (3), p. 732] we require such a specific condition to obtain the Wiener-Hopf integral equation. In (3), this specific condition could be avoided because there were two choices in formulating the problem. Here, in order to formulate the problem as a Wiener-Hopf integral equation, there is only one choice. Finally for $x \to \infty$, we do not require a specific form of $\phi(x, y)$ but merely an order condition. To this end we state that

$$\phi(x, y) = O[\exp(\theta x)], \qquad x \to \infty, \qquad \theta < \sqrt{(\rho_1^2/a^2) + k^2} \tag{2.7}$$

where ρ_1 is the smallest imaginary root of (2.6). The origin of the condition (2.7) will be described in the formulation of the problem.

For σ_1 sufficiently small and positive, $\varkappa^2$ is negative and here we shall only require order conditions on $\phi(x, y)$ for $|x| \to \infty$. Such an order condition is $\phi(x, y) = O[\exp(\theta |x|)]$ (1, 3) where $\theta < \sqrt{k^2 - (\rho_0/a^2)}$. For σ_1 negative a similar condition is in force. That is, we now require that $\phi(x, y) = O[\exp(\theta' |x|)]$ for $|x| \to \infty$ where $\theta' < \sqrt{k^2 + (\rho_0'^2/a^2)}$. Here ρ_n', $n = 0, 1, \cdots$ are the imaginary roots of (2.6), for $\sigma_1 < 0$.

3. The Green's Function and The Integral Equation. We desire a Green's function associated with the homogeneous equation (2.1). That is, we desire a solution of (2.1) which is finite everywhere save at $x = x'$, $y = y'$. At this point, $G(x, y, x', y')$, the Green's function has a logarithmic singularity, since it is a two dimensional one associated with eq. (2.1). Now as for the boundary conditions at $y = 0$ and $y = a$, we recognize that we have a certain degree of freedom. For example, if we take $\partial G/\partial y = 0$ at $y = 0$, we eliminate any contributions from the line $y = 0$ in the expression

$$G(\partial\phi/\partial y) - \phi(\partial G/\partial y) \tag{3.1}$$

Further for the line $y = a$, we can take $\partial G/\partial y = \sigma_1 G$ or $\partial G/\partial y = \sigma_2 G$. Let us take the first choice. Then for $x < 0$, the expression vanishes, while for $x > 0$ it becomes

$$(\sigma_2 - \sigma_1) G\phi$$

Such a Green's function has the expansion for $\varkappa$ real x

$$G(x, y, x', y') = \sum_{n=1}^{\infty} \left\{ \frac{\cos(\rho_n y/a) \cos(\rho_n y'/a)}{\sqrt{k^2 a^2 + \rho_n^2}\,(a^2\sigma_1^2 + \rho_n^2 - a\sigma_1)} \right\} \exp\left[-|x - x'| \sqrt{k^2 + (\rho_n^2/a^2)}\right](a^2\sigma_1^2 + \rho_n^2)$$

$$+ \frac{i(\rho_0^2 - a^2\sigma_1^2) \cosh(\rho_0 y/a) \cosh(\rho_0 y'/a) \exp[i\varkappa |x - x'|]}{a\varkappa(\rho_0^2 - a^2\sigma_1^2 + a\sigma_1)} \tag{3.2}$$

where the ρ_n's are the imaginary roots of (2.6). This G satisfies the Sommerfeld "Austrahlungsbedingung." But the presence of the term $\exp[i\varkappa \mid x - x' \mid]$ means finiteness at infinity and therefore (3.2) is not in a usable form $\mid x - x' \mid \to \infty$. However, if we add an appropriate solution of (2.1) to (3.2) we can retain the essential character of G and change the form of G for $\mid x - x' \mid \to \infty$. Such a solution is

$$-\frac{i(\rho_0^2 - a^2\sigma_1^2)}{a\varkappa(\rho_0^2 - a^2\sigma_1^2 + a\sigma_1)} \cosh\frac{\rho_0 y}{a} \cosh\frac{\rho_0 y'}{a} \exp[i\varkappa(x' - x)].$$

Then the modified Green's function has the properties that for $(x - x') \to \infty$, it is $O[\sin \varkappa(x - x')]$ while for $(x - x') \to -\infty$ it is $O[\exp(-\sqrt{k^2 + \rho_1^2/a^2})]$. This minor modification of G contains the clue to obtaining the Wiener-Hopf integral equation (4). Let us call the modified Green's function G'.

We have then that for $-L < x' < L'(L, L' > 0)$, and x, y interior to the strip $-L < x' < L', 0 \leq y \leq a$

$$\begin{aligned}\phi(x, y) &= \oint [G'(\partial\phi/\partial n') - \phi(\partial G'/\partial n')]\, ds' \\ &= [\alpha_1 \exp(i\varkappa x) + \beta_1 \exp(-i\varkappa x)] \cosh(\rho_0 y/a) \\ &\quad + (\sigma_2 - \sigma_1) \int_0^{L'} G'(x, y, x_1', a)\phi(x_1', a)\, dx' \\ &\quad + O[\exp\{(\theta - \sqrt{k^2 + (\rho_1^2/a^2)})L'\}] + O[\exp(-L\sqrt{k^2 + (\rho_1/a^2)})].\end{aligned}$$

Hence in the limit as L and L' become infinite, we have the Wiener-Hopf integral equation

$$\begin{aligned}\phi(x, a) &= \cosh\rho_0[\alpha, \exp(i\varkappa x) + \beta_1 \exp(-i\varkappa x)] \\ &\quad + (\sigma_2 - \sigma_1)\int_0^{\infty} G'(x, a, x_1', a)\phi(x_1', a)\, dx'.\end{aligned} \tag{3.3}$$

For $\varkappa^2$ negative, a minor change occurs in the Green's function. We have for $\sigma_1 > 0$

$$\begin{aligned}G^{(2)}(x_1 y_1 x_1' y') = \sum_1^{\infty} \Bigg\{ &\frac{\cos(\rho_n y/a)\cos(\rho_n y'/a)}{\sqrt{k^2a^2 + \rho_n^2}\,(a^2\sigma_1^2 + \rho_n^2 - a\sigma_1)} \\ &\exp[-\mid x - x' \mid \sqrt{k^2 + (\rho_n^2/a^2)}](a^2\sigma_1^2 + \rho_n^2)\Bigg\} \\ &+ \frac{\cosh(\rho_0 y/a)\cosh(\rho_0 y'/a)(\rho_0^2 - a^2\sigma_1^2)}{\sqrt{k^2a^2 - \rho_0^2}(a\sigma_1 + \rho_0^2 - a^2\sigma_1^2)} \\ &\exp[-\sqrt{k^2 - \rho_0^2/a^2}\,[x - x' \mid]\end{aligned}$$

Here the integral equation is

$$\phi(x, a) = (\sigma_2 - \sigma_1)\int_0^{\infty} G^{(2)}(x, a, x', a)\phi(x', a)\, dx'. \tag{3.4}$$

Finally for $\sigma_1 < 0$, $\varkappa^2 = -[k^2 + (\rho_0'^2/a^2)]$ and

$$G^{(3)}(x, y, x', y') = \sum_0^\infty \frac{\cos(\rho_n' y/a)\cos(\rho_n' y'/a)\exp[-|x-x'|\sqrt{k^2+(\rho_0')^2/a^2)}][a^2\sigma_1^2+\rho_n'^2]}{\sqrt{k^2a^2+\rho_n'^2}\,(a^2\sigma_1^2+\rho_n'^2-a\sigma_1)}$$

Here the integral equation is

$$\phi(x, a) = (\sigma_2 - \sigma_1)\int_0^\infty G^{(3)}(x, a, x', a)\phi(x', a)\,dx'. \tag{3.5}$$

4. The Fourier Transform Solution of Eqs. (3.3), (3.4) and (3.5). The actual solution of these equations follows along the lines described in (2, 3, 4) and we shall simply summarize the results here. We shall assume henceforth that $\varkappa'^2 = \nu_0^2/a^2 - k^2 > 0$. Here ν_0 is the real positive root of

$$\sigma_2 a\cos\nu - \nu\sinh\nu = 0$$

Then three cases arise as we have already noted.

1. $\sigma_1 > 0$, $\varkappa^2 > 0$. Here

$$\Phi(w) = \int_0^\infty \phi(x, a)\exp(-iwx)\,dx = \frac{\cosh\rho_0}{iK_-(w)}\left[\frac{\alpha_1 K_+(\varkappa)}{w-\varkappa} + \frac{\beta_1 K_+(-\varkappa)}{w+\varkappa}\right]$$

where

$$\frac{K_-(w)}{K_+(w)} = \frac{\sigma_2\cosh\gamma a - \gamma\sinh\gamma a}{\sigma_1\cosh\gamma a - \gamma\sinh\gamma a}, \qquad \gamma = \sqrt{k^2+w^2}$$

$$K_-(w) = \frac{\sigma_2\rho_0^2}{\sigma_1\nu_0^2}\frac{(\nu_0^2-\gamma^2a^2)}{(\rho_0^2-\gamma^2a^2)}L_-(w),$$

$$L_-(w) = \frac{1}{L_+(-w)} = \frac{\prod_1^\infty[\sqrt{1+(k^2a^2/\nu_n^2)}+iaw/\nu_n]\exp(-iaw/n\pi)}{\prod_1^\infty[\sqrt{1+(k^2a^2/\rho_n^2)}+iaw/\rho_n]\exp(-iaw/n\pi)},$$

and $L_+(w) = K_+(w)$

2. $\sigma_1 > 0$, $\varkappa^2 < 0$. Now

$$K_-(w) = -\frac{\sigma_2\rho_0(\nu_0^2-\gamma^2a^2)L_-(w)}{\sigma_1\nu_0^2[\sqrt{a^2k^2-\nu_0^2}+iaw]}$$

$$1/K_+(w) = \rho_0/L_+(w)[\sqrt{a^2k^2-\nu_0^2}-iaw]$$

and $L_-(w)$ and $L_+(w)$ have been defined in the previous paragraph.

3. $\sigma_1 < 0$. Now

$$K_-(w) = (\sigma_2/\sigma_1)L_-(w)[1-(\gamma^2a^2/\nu_0^2)]$$

and

$$K_+(w) = L_+(w)$$

Here $L_-(w)$ and $L_+(w)$ have already been defined. The lower index in the infinite

product depending upon the parameter σ_1 is now zero. For cases 2 and 3

$$\Phi(w) = C/K_-(w)$$

where C is an arbitrary constant.

For case 1 the following asymptotic forms arise. For $x \to -\infty$

$$\phi(x, y) = \cosh(\rho_0 y/a)[\alpha_1 \exp(ix\kappa) + \beta_1 \exp(-ix\kappa)]$$

while for $x \to \infty$

$$\phi(x, y) = \cosh(\nu_0 \rho/a)[\alpha_2 \exp(ix'x) + \beta_2 \exp(-ix'x)].$$

This follows from a residue calculation (4). We have

$$\alpha_2 = -\frac{\nu_0 \sigma_2(\sigma_2 - \sigma_1)\cosh \rho_0}{\kappa'(\sigma_2^2 a^2 - \sigma_2 a - \nu_0^2)\sinh \nu_0} \cdot \left[\frac{\alpha_1 K_+(\kappa)}{(\kappa' - \kappa)K_+(\kappa')} + \frac{\beta_1 K_+(-\kappa)}{(\kappa' + \kappa)K_+(\kappa')}\right], \tag{4.1}$$

$$\beta_2 = \frac{\nu_0 \sigma_2(\sigma_2 - \sigma_1)\cosh \rho_0}{\kappa'(\sigma_2^2 a^2 - \sigma_2 a - \nu_0^2)\sinh \nu_0} \cdot \left[\frac{-\alpha_1 K_+(\kappa)}{(\kappa' + \kappa)K_+(-\kappa')} + \frac{\beta_1 K_+(-\kappa)}{(\kappa - \kappa')K_+(-\kappa)}\right]. \tag{4.2}$$

That is, we have linear relations between the amplitudes of the waves travelling to the right and left in the left duct and those travelling in similar directions in the right duct. We employ the convention that the wave $\exp(i\delta x)$, $\delta > 0$ travels to the right while the wave $\exp(-i\delta x)$ travels to the left. Hence, for $\beta_2 = 0$ we have an incident wave and a reflected wave in the left duct and a transmitted wave in the right duct. The magnitude of the reflected wave is given by

$$|R_1|^2 = [(\kappa - \kappa')/(\kappa + \kappa')]^2,$$

while that of the transmitted wave in the right duct (not normalized) is given by

$$|t_1|^2 = \frac{4\kappa^2(\nu_0^2 - a^2\sigma_2^2)(\sigma_1^2 a^2 - \sigma_1 a - \rho_0^2)}{(\kappa + \kappa')^2(\rho_0^2 - a^2\sigma_1^2)(\sigma_2^2 a^2 - \sigma_2 a - \nu_0^2)}$$

If instead of β_2 being zero, α_1 were zero, we would have a reflected wave in the right duct and a transmitted wave in the left duct. Now the magnitude of the reflected wave in the left duct is given by

$$|R_2|^2 = [(\kappa - \kappa')/(\kappa + \kappa')]^2$$

while that of the transmitted wave (not normalized) is

$$|t_2|^2 = \frac{4\kappa'^2(\rho_0^2 - a^2\sigma_1^2)(\sigma_2^2 a^2 - \sigma_2 a - \nu_0^2)}{(\kappa + \kappa')^2(\nu_0^2 - a^2\sigma_2^2)(\sigma_1^2 a^2 - \sigma_1 a - \rho_0^2)}.$$

Consider now, the magnitudes of α_1, α_2, β_1 and β_2. Of these four magnitudes, only three are independent. For since (4.1) and (4.2) are solutions of $\nabla^2\phi - k^2\phi =$

0, which satisfy the boundary conditions on the various segments, we have (4) (the asterisk denotes complex conjugate)

$$(4.3) \qquad \iint [\phi \nabla^2 \phi^* - \phi^* \nabla^2 \phi]\, dA = 0 = \oint [\phi(\partial\phi^*/\partial n) - \phi^*(\partial\phi/\partial n)]\, ds.$$

This implies a relation between the four magnitudes. When (4.3) is simplified, we get

$$(4.4) \qquad \frac{\varkappa[\,|\,\alpha_1\,|^2 - |\,\beta_1\,|^2][\rho_0^2 - a^2\sigma_1^2 - a\sigma_1]}{\rho_0^2 - a^2\sigma_1^2} = \frac{\varkappa'[\,|\,\alpha_2\,|^2 - |\,\beta_2\,|^2][\nu_0^2 - a^2\sigma_2^2 + a\sigma_2]}{\nu_0^2 - a^2\sigma_2^2}.$$

The expressions we have for $|\,\alpha_1\,|$, $|\,\beta_1\,|$, $|\,\alpha_2\,|$ and $|\,\beta_2\,|$ satisfy this relation (4.4). This identity provides us with a means of normalizing $|\,R_1\,|$, $|\,R_2\,|$, $|\,t_1\,|$ and $|\,t_2\,|$. For example, if $\beta_2 = 0$ (transmission to the right), we have

$$1 - |\,R_1\,|^2 = \frac{\varkappa'\,|\,t_1\,|^2(\rho_0^2 - a^2\sigma_1^2)(\nu_0^2 - a^2\sigma_2^2 + a\sigma_2)}{\varkappa(\nu_0^2 - a^2\sigma_2^2)(\rho_0^2 - a^2\sigma_1^2 + a\sigma_1)} = |\,T_1\,|^2,$$

where T_1 is the normalized transmission coefficient and is simply

$$|\,T_1\,|^2 = 4\varkappa\varkappa'/(\varkappa + \varkappa')^2$$

In a similar fashion we find that T_2, the normalized transmission coefficient in the left duct, is given by

$$|\,T_2\,|^2 = 4\varkappa\varkappa'/(\varkappa + \varkappa')^2.$$

It is clear that $|\,R_1\,|$ and $|\,R_2\,|$ are automatically normalized.

Cases 2 and 3 follow as in (3). In reference to (3) the following comment may be made. A Green's function was used which satisfies the boundary conditions that $\partial G/\partial y = 0$ at $y = 0$ and $y = a$. This was done so that only an order condition would be required for $x \to \infty$ to formulate the integral equation. Actually, it would have been equally good to use a Green's function similar to G' used in sec. 3 of this note, and to invoke a precise asymptotic boundary condition on one side as we have done here.

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THE STABILITY OF INFINITE DIFFERENTIAL SYSTEMS ASSOCIATED WITH VORTEX STREETS

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Introduction. The considerations of the present paper center about a k-fold infinite linear system of differential equations of the type

$$\dot{\zeta}_n = \sum_{m=-\infty}^{\infty} A_{nm}\zeta_m, \qquad (\cdot = d/dt). \tag{0.1}$$

The A_{nm} $(n,m = 0, \pm 1, \pm 2, \cdots)$ are k by k constant matrices with the particular structure $A_{nm} = A_{n-m}$, and the ζ_n $(n = 0, \pm 1, \pm 2, \cdots)$ are vectors of one column and k rows. If A is defined to be the infinite matrix whose elements are the matrices A_{nm}, and ζ is the infinite vector with elements ζ_n, then (0.1) may be written simply as

$$\dot{\zeta} = A\zeta. \tag{0.2}$$

For the case $k = 1$, Toeplitz called A a Laurent matrix since the study of such A is intimately related to Laurent series with the A_p as coefficients (cf. [8]). Under the assumption that the elements of the matrices A_p are Fourier constants of continuous functions on an interval $0 \leqq \mu \leqq 1$, we construct in Section 1 the unique analytic solution $\zeta = \zeta(t)$ of (0.2) corresponding to any initial vector $\zeta^0 = (\zeta_n^{0i})$, $i = 1, \cdots, k$, $n = 0, \pm 1, \pm 2, \cdots$, which is in ordinary Hilbert space, that is

$$\| \zeta^0 \| = \left(\sum_{n=-\infty}^{\infty} \sum_{i=1}^{k} | \zeta_n^{0i} |^2 \right)^{1/2} < \infty .$$

In Section 2 we study the behavior of the solution ζ as $t \to +\infty$. The system (0.2) is said to be *stable*, with respect to the set of all initial vectors ζ^0 for which $\| \zeta^0 \| < \infty$, if and only if $\limsup \| \zeta \| < \infty$, $t \to +\infty$, holds for every solution corresponding to such an initial vector. We give necessary, and sufficient, conditions for stability.

The principal impetus in this work has been the possible application of such results to the physical problem that von Kármán considered in connection with the stability of parallel rows of vortices [1], [2]. In fact Wintner, in a short note some time ago [10], suggested that his work on equations of the type (0.1) with $k = 1$ (cf. [9]) could be amplified to treat the von Kármán problem. In Section 3 we consider the *linear* problem of the first approximation resulting from perturbing two infinite rows of vortices originally situated at the points in the (x, y) - plane $(2nb + c, a)$, $(2nb - c, -a)$, $n = 0, \pm 1, \pm 2, \cdots$; see Figure 1. The linear (infinite) system of differential equations resulting is of the type (0.2), and if $r = a/b$, $q = c/b$, it is shown that the system is unstable when $q \neq \frac{1}{2}$. In fact there exist initial vectors ζ^0 such that $\| \zeta(t) \| \to +\infty$ at least as fast as some exponential $e^{\alpha t}$, $\alpha > 0$. For the case $q = \frac{1}{2}$ we construct the solution

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explicitly and show that a necessary condition for stability is that $\cosh^2 \pi r = 2$. This is the criterion obtained by von Kármán. However, contrary to previous beliefs, we prove the result that the linear system is *not stable* when $q = \frac{1}{2}$ and $\cosh^2 \pi r = 2$. As a matter of fact we show that there exist initial perturbation vectors ζ^0 with arbitrarily small magnitudes $\| \zeta^0 \|$, but for which the corresponding solutions ζ are such that $\| \zeta \| \to +\infty$, as $t \to +\infty$. The example given to

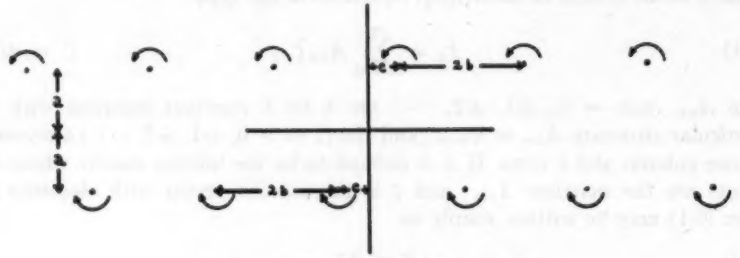

FIGURE 1. Von Kármán Vortex Street

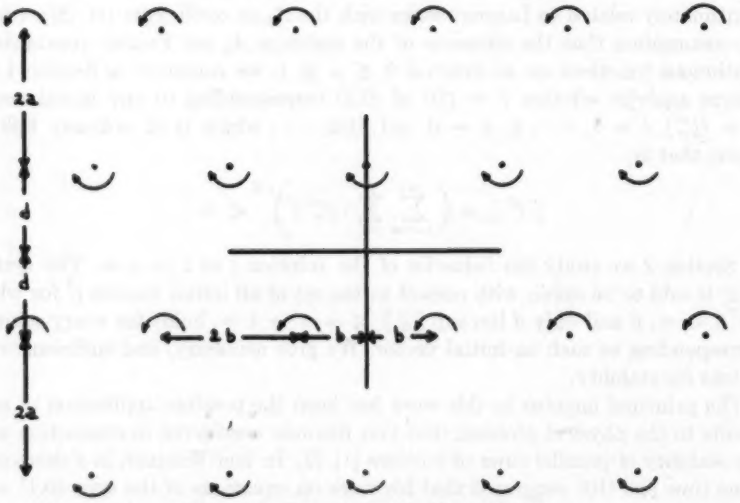

FIGURE 2. Type A Double Vortex Street

illustrate this shows that for any β, $0 < \beta < 2$, there exists a ζ^0 such that $\| \zeta(t) \| \to \infty$ faster than t^β, as $t \to +\infty$.

If the set of initial vectors is restricted to a certain subclass S, obtained by imposing certain symmetry relations, then it is shown that the linear system *is* stable with respect to the set S if $q = \frac{1}{2}$ and $\cosh^2 \pi r = 2$. It is not known what characterizes the set of all initial vectors ζ^0, $\| \zeta^0 \| < \infty$, for which the system is stable.

We compare our results with those of von Kármán. It is easy to see that the solutions obtained by using his special assumptions are not stable in our sense, *nor even according to his definition of stability.* Using the von Kármán type of perturbation (which involves using initial vectors ζ^0 such that $\| \zeta^0 \| = \infty$), it can be shown that there exist initial vectors such that the corresponding solution vectors have components which tend to infinity as the first power of t, as $t \to +\infty$. Such solutions, it seems, were not observed by von Kármán because he did not discuss the possibility that a zero characteristic root of the matrix determining the solution of his system might be associated with *non-simple* elementary divisors.

We have also considered the problem of the stability of four rows of vortices. The two configurations investigated are pictured in Figures 2 and 3. In Section

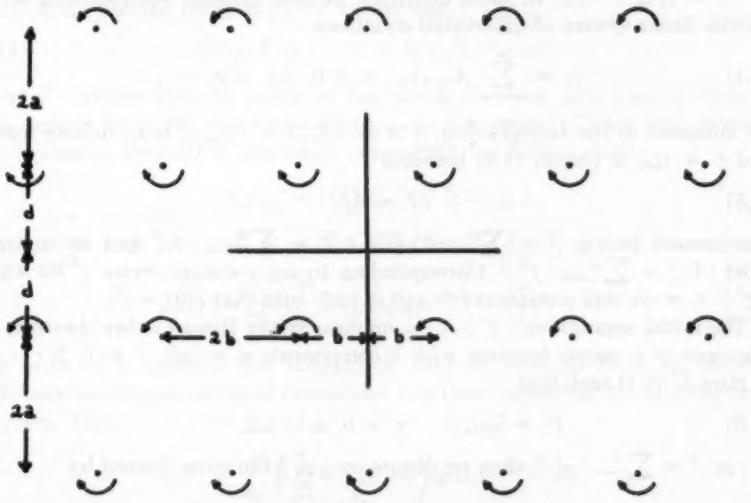

FIGURE 3. Type B Double Vortex Street

4 we give the explicit solutions of the linear systems corresponding to these, and obtain necessary conditions for stability.

It should be emphasized that the results of Sections 3 and 4 apply only to the *linear* problem which is obtained by ignoring higher order terms in the non-linear system of differential equations which describe the physical situation. Consequently any conclusions about the physical problem must be viewed in this light.

1. Solution of the Infinite Linear System. Let there be given, for each $p = 0, \pm 1, \pm 2, \cdots$, a k by k matrix of constants $A_p = (a_p^{ij})$, $i, j = 1, 2, \cdots, k$. Suppose that there exists a matrix $F = (f^{ij})$, $i, j = 1, 2, \cdots, k$, of continuous functions on $0 \leqq \mu \leqq 1$ which have the constants a_p^{ij} as the p-th Fourier coefficients, that is,

$$F(\mu) \sim \sum_{p=-\infty}^{\infty} A_p e^{2\pi i p\mu}. \tag{1.1}$$

If the p-th Fourier constant of an integrable function α on $0 \leqq \mu \leqq 1$ is denoted by

$$[\alpha]_p = \int_0^1 \alpha(\mu) e^{-2\pi i p\mu}\, d\mu, \tag{1.2}$$

then (1.1) may be written simply as

$$[F]_p = A_p, \tag{1.3}$$

where $[F]_p$ is the matrix with elements $[f^{ij}]_p$. For any $n = 0, \pm 1, \pm 2, \cdots$ denote by ζ_n the vector represented as a one column matrix with k elements ζ_n^i, $i = 1, 2, \cdots, k$. In these notations we now consider the following k-fold infinite linear system of differential equations

$$\dot{\zeta}_n = \sum_{m=-\infty}^{\infty} A_{n-m}\zeta_m, \; n = 0, \pm 1, \pm 2, \cdots. \tag{1.4}$$

As indicated in the Introduction, if $A = (A_{nm}) = (A_{n-m})$ is an infinite matrix and $\zeta = (\zeta_n)$ a vector, (1.4) becomes

$$\dot{\zeta} = A\zeta. \tag{1.5}$$

For vectors let $|\zeta_n|^2 = \sum_{i=1}^k |\zeta_n^i|^2$, $\|\zeta\|^2 = \sum_{n=-\infty}^{\infty} |\zeta_n|^2$ and for matrices F let $|F|^2 = \sum_{i,j=1}^k |f^{ij}|^2$. Corresponding to any constant vector ζ^0 for which $\|\zeta^0\| < \infty$ we seek a solution $\zeta = \zeta(t)$ of (1.5) such that $\zeta(0) = \zeta^0$.

The initial assumption $\|\zeta^0\| < \infty$ implies, by the Riesz-Fischer theorem, the existence of a vector function with k components $\varphi_0 = (\varphi_0^i)$, $i = 1, 2, \cdots, k$, of class $L^2(0, 1)$ such that

$$\zeta_n^0 = [\varphi_0]_n, \qquad n = 0, \pm 1, \pm 2, \cdots. \tag{1.6}$$

If $|\varphi_0|^2 = \sum_{i=1}^k |\varphi_0^i|^2$, then we denote by $\|\varphi_0\|$ the norm defined by

$$\|\varphi_0\|^2 = \int_0^1 |\varphi_0|^2\, d\mu.$$

Thus $\|\varphi_0\| = \|\zeta^0\|$. Using φ_0 as a start try for a solution to (1.5) the formal power series

$$\zeta_n(t) = \sum_{r=0}^{\infty} [\varphi_r]_n t^r/r!, \tag{1.7}$$

where the functions φ_r, $r \neq 0$, are as yet undetermined functions of class $L^2(0, 1)$. Using (1.3) substitute (1.7) into (1.4). This results in

$$\sum_{r=0}^{\infty} [\varphi_{r+1}]_n t^r/r! = \sum_{m=-\infty}^{\infty} [F]_{n-m} \sum_{r=0}^{\infty} [\varphi_r]_m t^r/r!. \tag{1.8}$$

From the Parseval theorem, if g_1 and g_2 are two functions of class $L^2(0, 1)$, the p-th Fourier constant of the product $g_1 g_2$ is given by the convolution

$$[g_1 g_2]_p = \sum_{m=-\infty}^{\infty} [g_1]_{p-m}[g_2]_m . \tag{1.9}$$

Applying this remark to (1.8), after interchanging the orders of summation and comparing coefficients of t^r, we obtain formally

$$[\varphi_{r+1}]_n = [F\varphi_r]_n , \qquad n = 0, \pm 1, \pm 2, \cdots , r = 0, 1, 2, \cdots . \tag{1.10}$$

Now (1.10) will certainly hold if

$$\varphi_{r+1} = F\varphi_r , \qquad r = 0, 1, 2, \cdots . \tag{1.11}$$

Since φ_0 was defined by the initial condition (1.6), this relation is a recurrence formula for the vectors φ_r. Because F was assumed to be a matrix of continuous functions on $0 \leqq \mu \leqq 1$, and $\varphi_0 \in L^2(0, 1)$, it follows from (1.11) that all $\varphi_r \in L^2(0, 1)$. Solving for φ_r in (1.11) we see that

$$\varphi_r = F^r\varphi_0 , \qquad r = 0, 1, 2, \cdots , \tag{1.12}$$

where F^r denotes the r-th power of the matrix F, F^0 = unit matrix. Thus if φ_r is defined by (1.12), the coefficients $[\varphi_r]_n$ in the series (1.7) are well defined, and the series for $\zeta_n(t)$ is absolutely convergent for finite t since

$$| [\varphi_r]_n | = | [F^r\varphi_0]_n | \leqq M^r \| \varphi_0 \| ,$$

where $M = \max | F |$ on $0 \leqq \mu \leqq 1$. Substituting (1.12) into (1.7) results in

$$\zeta_n(t) = \sum_{r=0}^{\infty} \left(\int_0^1 F^r(\mu)\varphi_0(\mu)e^{-2\pi i n\mu}\, d\mu \right) t^r/r!.$$

The order of integration and summation may be changed since we have a uniformly convergent series of continuous functions multiplied by an integrable function. Thus

$$\zeta_n(t) = \int_0^1 \left(\sum_{r=0}^{\infty} F^r(\mu)t^r/r! \right) \varphi_0(\mu)e^{-2\pi i n\mu}\, d\mu,$$

which is the same as

$$\zeta_n(t) = \int_0^1 e^{F(\mu)t}\varphi_0(\mu)e^{-2\pi i n\mu}\, d\mu. \tag{1.13}$$

From (1.13) it is easy to see that ζ_n satisfies the differential equation (1.4). From (1.6) we have

$$\varphi_0(\mu) \sim \sum_{m=-\infty}^{\infty} \zeta_m^0 e^{2\pi i m\mu},$$

and if this is placed into (1.13), and the result integrated term by term, there results

$$\zeta_n(t) = \sum_{m=-\infty}^{\infty} R_{n-m}(t)\zeta_m^0 , \tag{1.14}$$

where

$$R_p(t) = [e^{Ft}]_p . \tag{1.15}$$

Therefore, if $R(t)$ is the infinite matrix with elements $R_{nm}(t) = R_{n-m}(t)$, then a solution ζ of (1.5) corresponding to any initial condition $\zeta(0) = \zeta^0$, $\| \zeta^0 \| < \infty$, is given by

$$\zeta(t) = R(t)\zeta^0. \tag{1.16}$$

From the representation (1.13) it follows that $\zeta_n(t) = [e^{Ft}\varphi_0]_n$, and by the Parseval equality $\| \zeta(t) \| = \| e^{Ft}\varphi_0 \|$. This shows, in particular, that $\zeta(t)$ is in Hilbert space for all finite t. The classical proof of uniqueness for finite systems of differential equations satisfying a Lipschitz condition applies without change to infinite linear systems having a coefficient matrix which is bounded. The coefficient matrix A (cf. (1.5)) is certainly bounded. If $\eta = (\eta_n)$ is any vector for which $\| \eta \| < \infty$, then there exists a vector function $\psi = \psi(\mu)$ on $0 \leqq \mu \leqq 1$, $\| \psi \| < \infty$, such that $[\psi]_n = \eta_n$, and $\| \eta \| = \| \psi \|$. But from (1.3) it follows that the n-th component of $A\eta$ is given by

$$(A\eta)_n = \sum_{m=-\infty}^{\infty} [F]_{n-m}[\psi]_m = [F\psi]_n ,$$

and hence, by the Parseval formula, $\| A\eta \| = \| F\psi \| \leqq M \| \psi \| = M \| \eta \|$, which proves A is bounded. Thus the analytic solution (1.16) of (1.5) is unique.

We summarize the above results in the following theorem.

THEOREM 1. *Let $\{A_p\}$, $p = 0, \pm 1, \pm 2, \cdots$, be a sequence of k by k matrices which are Fourier constants for a matrix F of continuous functions on $0 \leqq \mu \leqq 1$, and suppose A is the infinite matrix with elements $A_{nm} = A_{n-m}$. Let $\zeta = (\zeta_n)$ be an infinite vector whose elements ζ_n are k-dimensional vectors. Then, for every initial vector ζ^0 for which $\| \zeta^0 \| < \infty$, there exists a unique analytic solution of the infinite linear system*

$$\dot{\zeta} = A\zeta, \qquad \zeta(0) = \zeta^0,$$

which has the form

$$\zeta(t) = R(t)\zeta^0,$$

where

$$R(t) = (R_{n-m}(t)) = \left(\int_0^1 e^{F(\mu)t} e^{-2\pi i(n-m)\mu} \, d\mu \right).$$

Further, for every t, $0 \leqq t < \infty$, $\| \zeta(t) \| < \infty$.

2. Behavior of Solutions at Infinity. Since $\| \zeta(t) \| = \| e^{Ft}\varphi_0 \|$ it is clear that the behavior of $\| \zeta(t) \|$ as $t \to +\infty$ is entirely dependent on the nature of the *finite* matrix of continuous functions $F(\mu)$. Let $\lambda_i(\mu)$, $i = 1, \cdots, k$, denote, in some order, the characteristic roots of $F(\mu)$, and let $R\lambda_i(\mu)$ denote the corresponding real parts. The situation may then be summarized as follows.

THEOREM 2. (a) *If* $\lim \sup \| \zeta(t) \| < \infty$, $t \to +\infty$, *for every* $\varphi_0 \in L^2(0, 1)$, *then* $R\lambda_i(\mu) \leqq 0$ *holds for* $0 \leqq \mu \leqq 1$, *and* $i = 1, \cdots, k$.

(b) *If* $R\lambda_i(\mu) < 0$ *for all* μ, $0 \leqq \mu \leqq 1$, *and all* i, $i = 1, \cdots, k$, *then, for every* $\varphi_0 \in L^2(0, 1)$, $\| \zeta(t) \| = O(e^{-ct})$, $t \to +\infty$, *where* c *is some positive constant.*

(c) *Suppose* $R\lambda_i(\mu) \leqq 0$ *for all* μ, $0 \leqq \mu \leqq 1$, *and all* i, $i = 1, \cdots, k$, *and that* $R\lambda_i(\mu) = 0$ *for some* μ *and* i. *If all* $\lambda_i(\mu)$ *are distinct then, for every* $\varphi_0 \in L^2(0, 1)$, $\| \zeta(t) \| = O(1)$, $t \to +\infty$.

Remark. If $F(\mu)$ has a multiple root $\lambda_i(\mu)$ at just one point which corresponds to non-simple elementary divisors, and $R\lambda_i(\mu) = 0$ there, it can happen that $\| \zeta(t) \| \to +\infty$, $t \to +\infty$ for some $\varphi_0 \in L^2(0, 1)$. As a matter of fact this is precisely the situation which arises in the application considered in Theorem 4 below. It is not known whether there exist $F(\mu)$ $(k > 1)$ with this property, but for which $\lim \sup \| \zeta(t) \| < \infty$, $t \to +\infty$, for all $\varphi_0 \in L^2(0, 1)$.

Proof of (a). Under the hypothesis,

$$\lim_{t \to +\infty} \sup \int_0^1 | e^{F(\mu)t} |^2 \, d\mu < \infty. \tag{2.1}$$

This follows from the fact that the unit vectors $\varphi_{0j} = (\varphi_{0j}^i)$, $\varphi_{0j}^i = \delta_{ij}$, are of class $L^2(0, 1)$, and for φ_{0j}

$$\| \zeta(t) \|^2 = \int_0^1 \sum_{i=1}^k | (e^{F(\mu)t})_{ij} |^2 \, d\mu,$$

where $(e^{F(\mu)t})_{ij}$ is the element in the i-th row and j-th column of $e^{F(\mu)t}$. Now $| e^{F(\mu)t} |^2 = tr(e^{F^*(\mu)t} e^{F(\mu)t})$, where F^* is the conjugate transpose, and tr = trace, and it is in general true (cf. e.g. [6]) that

$$tr(e^{F^*(\mu)t} e^{F(\mu)t}) \geqq \sum_{i=1}^k | e^{\lambda_i(\mu)t} |^2 = \sum_{i=1}^k e^{2R\lambda_i(\mu)t}. \tag{2.2}$$

The quantity on the right of (2.2) is integrable over $0 \leqq \mu \leqq 1$, for the $\lambda_i(\mu)$ are the roots of an algebraic equation $\det(\lambda E_k - F(\mu)) = 0$ ($E_k = k$ by k unit matrix) whose coefficients are continuous functions of μ on $0 \leqq \mu \leqq 1$. Consequently, from (2.1) and (2.2) we have

$$\lim_{t \to +\infty} \sup \int_0^1 \sum_{i=1}^k e^{2R\lambda_i(\mu)t} \, d\mu < \infty, \tag{2.3}$$

and therefore, by virtue of the continuity of the $\lambda_i(\mu)$, it must be true that $R\lambda_i(\mu) \leqq 0$ for all μ, $0 \leqq \mu \leqq 1$, and $i = 1, \cdots, k$.

From the proof it can be seen that if $R\lambda_i(\mu) > 0$ for some i and μ then there exists a $\varphi_0 \in L^2(0, 1)$ such that $\| \zeta(t) \| \to +\infty$ at least as fast as some exponential e^{ct}, $c > 0$.

Proof of (b). By virtue of the hypothesis, and because of the continuity of the $R\lambda_i(\mu)$, there exists a positive constant c such that $R\lambda_i(\mu) \leqq -c$ for all μ, $0 \leqq \mu \leqq 1$, and all i, $i = 1, \cdots, k$.

First* it will be shown that for every particular μ_0, $0 < \mu_0 < 1$, there exists

* This argument was suggested by Professor N. Levinson.

an open interval $I(\mu_0)$ in $0 \leqq \mu \leqq 1$ containing μ_0 in its interior, and a constant $K(\mu_0)$, such that

$$| e^{F(\mu)t} |^2 \leqq K(\mu_0)e^{-\alpha t} \tag{2.4}$$

holds for all μ in $I(\mu_0)$. In order to prove this we note that for a fixed μ, the columns of $e^{F(\mu)t}$ represent k linearly independent solution vectors, $x_1(t, \mu), \cdots, x_k(t, \mu)$, of the *finite* linear system

$$\dot{x} = F(\mu)x, \tag{2.5}$$

that is, $e^{F(\mu)t}$ is a fundamental matrix for (2.5). It is that particular one which reduces to the unit matrix at $t = 0$. Therefore

$$| e^{F(\mu)t} |^2 = \sum_{i=1}^{k} | x_i(t, \mu) |^2, \tag{2.6}$$

and consequently we appraise $| x_i(t, \mu) |^2$. Note that $| x_i(0, \mu) |^2 = 1$.

To do this let $T(\mu_0)$ be a non-singular matrix such that $J(\mu_0) = T^{-1}(\mu_0)F(\mu_0)T(\mu_0)$ has the normal form

$$J(\mu_0) = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & 0 \\ 0 & \cdots & 0 & J_q \end{pmatrix}, \tag{2.7}$$

where the matrices J_i are of the form

$$J_i = \begin{pmatrix} \lambda_j(\mu_0) & 0 & \cdots & 0 \\ \alpha & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & 0 \\ 0 & \cdots & \alpha & \lambda_{j+p_i}(\mu_0) \end{pmatrix}, \lambda_j(\mu_0) = \cdots = \lambda_{j+p_i}(\mu_0), \alpha > 0. \tag{2.8}$$

It is understood that any (or all) of the J_i may be matrices of one row and column. Since $F(\mu)$ is continuous at μ_0, the matrix $G(\mu, \mu_0) = T^{-1}(\mu_0)F(\mu)T(\mu_0)$ will be also. Hence we write

$$G = G(\mu, \mu_0) = J(\mu_0) + \Delta(\mu), \tag{2.9}$$

where each element of $\Delta(\mu)$ tends to zero as $\mu \to \mu_0$.

Now consider the system

$$\dot{y} = Gy. \tag{2.10}$$

From (2.10) it follows that $y^*\dot{y} = y^*Gy$ and $\dot{y}^*y = y^*G^*y$, and by addition

$$(d/dt)\,|\,y\,|^2 = (d/dt)(y^*y) = y^*(G + G^*)y. \tag{2.11}$$

But from (2.9), $G + G^* = J(\mu_0) + J^*(\mu_0) + \Delta(\mu) + \Delta^*(\mu)$, and $J(\mu_0) + J^*(\mu_0)$ is a matrix with diagonal elements $2R\lambda_i(\mu_0)$ and at most $2(k-1)$ α's off the diagonal. Since $2R\lambda_i(\mu_0) \leqq -2c$, it follows from a straightforward calculation that

$$|\,y^*(G + G^*)y - (-2cy^*y)\,| \leqq ([2(k-1)]^{1/2}\,\alpha + |\,\Delta(\mu) + \Delta^*(\mu)\,|\,)y^*y. \tag{2.12}$$

Now α can be made small by a proper choice of $T(\mu_0)$, and $|\,\Delta(\mu) + \Delta^*(\mu)\,| \to 0$ as $\mu \to \mu_0$. Therefore, there exists a $\delta > 0$ and a $T(\mu_0)$ such that

$$[2(k-1)]^{1/2}\,\alpha + |\,\Delta(\mu) + \Delta^*(\mu)\,| < c \tag{2.13}$$

holds on $I(\mu_0)$: $|\,\mu - \mu_0\,| < \delta$. Hence, from (2.12),

$$y^*(G + G^*)y \leqq -cy^*y, \qquad \mu \text{ on } I(\mu_0), \tag{2.14}$$

and on account of (2.11),

$$(d/dt)\,|\,y\,|^2 \leqq -c\,|\,y\,|^2, \qquad c > 0, \qquad \mu \text{ on } I(\mu_0). \tag{2.15}$$

From (2.15) it now follows by integrating that, for any non-zero solution $y(t, \mu)$ of (2.10),

$$|\,y(t, \mu)\,|^2 \leqq |\,y(0, \mu)\,|^2 e^{-ct}, \qquad \mu \text{ on } I(\mu_0). \tag{2.16}$$

If $y(t, \mu)$ is a solution of (2.10), then $x(t, \mu) = T(\mu_0)y(t, \mu)$ is a solution of (2.5), and conversely. We are interested in those solutions $x(t, \mu)$ of (2.5) for which $|\,x(0, \mu)\,|^2 = 1$. For these $|\,y(0, \mu)\,|^2 \leqq |\,T^{-1}(\mu_0)\,|^2$, and hence, from (2.16),

$$|\,y(t, \mu)\,|^2 \leqq |\,T^{-1}(\mu_0)\,|^2 e^{-ct}, \qquad \mu \text{ on } I(\mu_0). \tag{2.17}$$

If $K(\mu_0) = k\,|\,T(\mu_0)\,|^2\,|\,T^{-1}(\mu_0)\,|^2$, then

$$|\,x(t, \mu)\,|^2 \leqq (K(\mu_0)/k)e^{-ct}, \qquad \mu \text{ on } I(\mu_0), \tag{2.18}$$

and if (2.18) is applied to each of the k linearly independent solution vectors $x_1, \cdots, x_k$ which comprise the columns of $e^{F(\mu)t}$, then (2.4) obtains. Clearly open intervals $I(0)$ and $I(1)$ can be constructed containing $\mu = 0$ and $\mu = 1$ which have the property (2.4).

We apply the Heine-Borel theorem to the open covering $\{I(\mu_0)\}$ and obtain a finite sub-covering. Let these be denoted by $I_1, \cdots, I_s$, and let the associated constants be $K_1, \cdots, K_s$. Then, if $K = \max K_i$, $i = 1, \cdots, s$, it follows that

$$|\,e^{F(\mu)t}\,|^2 \leqq Ke^{-ct} \tag{2.19}$$

holds for *all* μ, $0 \leqq \mu \leqq 1$. From the fact that $\|\,\zeta(t)\,\|^2 = \|\,e^{Ft}\varphi_0\,\|^2$ and since $|\,e^{Ft}\varphi_0\,|^2 \leqq |\,e^{Ft}\,|^2\,|\,\varphi_0\,|^2$, it follows from (2.19) that

$$\|\,\zeta(t)\,\|^2 \leqq Ke^{-ct}\,\|\,\varphi_0\,\|^2.$$

Hence (b) is proved for the constant $c/2$.

Proof of (c). Since all $\lambda_i(\mu)$ are distinct (for all μ, $0 \leqq \mu \leqq 1$), it follows that

there exists a non-singular matrix $T = T(\mu)$ such that $F(\mu)T(\mu) = T(\mu)J(\mu)$ where $J(\mu)$ is a *diagonal* matrix with the characteristic roots $\lambda_i(\mu)$ along the diagonal. If the column vectors of T are denoted by $v_1, \cdots, v_k$, then we have $(\lambda_i E - F)v_i = 0$, $i = 1, \cdots, k$. Clearly the v_i can be so chosen that $|v_i| = 1$. Also, since $\lambda_i(\mu)E - F(\mu)$ is a continuous matrix in μ, $v_i(\mu)$ is continuous in μ. This implies that $T(\mu)$ is continuous and hence bounded on $0 \leqq \mu \leqq 1$. Now $T^{-1}(\mu)$ is also bounded on $0 \leqq \mu \leqq 1$, for otherwise there would exist a sequence μ_n, $n = 1, 2, \cdots$, such that $\mu_n \to \mu_0$ for some μ_0, $0 \leqq \mu_0 \leqq 1$, and $\det T(\mu_n) \to 0$, $n \to \infty$. But $\det T(\mu)$ is continuous, and therefore $\det T(\mu_0) = 0$. This contradicts the fact that the columns of $T(\mu_0)$ are the unit vectors $v_i(\mu_0)$ which are linearly independent, for they are characteristic vectors corresponding to distinct characteristic roots of $F(\mu_0)$.

Now $F = TJT^{-1}$ implies that

$$e^{Ft} = e^{TJT^{-1}t} = Te^{Jt}T^{-1},$$

and thus $|e^{Ft}|^2 \leqq K|e^{Jt}|^2 = K\sum_{i=1}^{k} e^{2R\lambda_i t}$, where K is a constant independent of μ and t. But $R\lambda_i \leqq 0$ implies that $|e^{Ft}|^2 \leqq Kk$, and hence $\|\zeta(t)\|^2 \leqq Kk\|\varphi_0\|^2$, which proves (c).

3. The von Kármán Vortex Streets. We now apply the results of the preceding two sections to the physical problems considered by von Kármán and Rosenhead (cf. [1], [2], [5]). In two dimensions (x, y) consider a system consisting of an infinite row of vortex filaments, all of equal strength $\Gamma > 0$, situated at the points

$$(2nb + c, a), \qquad n = 0, \pm 1, \pm 2, \cdots, \tag{3.1}$$

and another infinite row of vortices, each of strength $-\Gamma$, located at the points (see Figure 1)

$$(2nb - c, -a), \qquad n = 0, \pm 1, \pm 2, \cdots. \tag{3.2}$$

Now consider the vortices at (3.1) displaced to

$$(2nb + c + x_n^1, a + y_n^1), \qquad n = 0, \pm 1, \pm 2, \cdots, \tag{3.1'}$$

and the vortices initially at (3.2) displaced to

$$(2nb - c + x_n^2, -a + y_n^2), \qquad n = 0, \pm 1, \pm 2, \cdots. \tag{3.2'}$$

The x-component of velocity of the n-th vortex of the upper row due to the motion of the other vortices is then given by (cf. e.g., [3], p. 219)

$$\frac{2\pi}{\Gamma}(U + \dot{x}_n^1) = -\sum_m{}^{*} \frac{(y_n^1 - y_m^1)}{(r_{nm}^1)^2} + \sum_m \frac{(y_n^1 - y_m^2 + 2a)}{(r_{nm}^2)^2}, \tag{3.3}$$

where

$$(3.4)\qquad \begin{cases}(r^1_{nm})^2 = [(x^1_n - x^1_m) + 2(n-m)b]^2 + [y^1_n - y^1_m]^2, \\ (r^2_{nm})^2 = [(x^1_n - x^2_m) + 2(n-m)b + 2c]^2 + [y^1_n - y^2_m + 2a]^2. \end{cases}$$

In (3.3) U represents the translational velocity of the vortex system in the x-direction, $\sum_m^*$ denotes summation over all integral m ($m = -\infty$ to $m = +\infty$) *excluding* $m = n$, $\sum_m$ denotes summation over all integral m. The corresponding equation for the y-component of velocity is

$$(3.5)\qquad \frac{2\pi}{\Gamma}(V + \dot{y}^1_n) = \sum_m^* \frac{(x^1_n - x^1_m + 2(n-m)b)}{(r^1_{nm})^2} - \sum_m \frac{(x^1_n - x^2_m + 2(n-m)b + 2c)}{(r^2_{nm})^2},$$

where V is the velocity of the vortex system in the y-direction. By changing the sign of Γ, a, c, and interchanging superscripts 1 and 2, the equations for the motion of the n-th vortex in the lower row may be obtained. What thus results is an *infinite, non-linear*, system of simultaneous differential equations in which every unknown occurs in every equation.

We shall consider the linear equations which result from (3.3), (3.5), and those for the lower row, when terms of the second and higher orders in the displacements x^1_n, y^1_n, x^2_n, y^2_n are neglected. For example, the linear equation corresponding to (3.3) is

$$(3.6)\qquad \begin{aligned}\frac{2\pi}{\Gamma}(U + \dot{x}^1_n) = {} & \sum_m \frac{a}{2[a^2 + ((n-m)b + c)^2]} - \sum_m^* \frac{y^1_n - y^1_m}{4b^2(n-m)^2} \\ & + \sum_m \frac{[((n-m)b + c)^2 - a^2]}{4[a^2 + ((n-m)b + c)^2]^2}(y^1_n - y^2_m) \\ & - \sum_m \frac{a((n-m)b + c)}{2[a^2 + ((n-m)b + c)^2]^2}(x^1_n - x^2_m).\end{aligned}$$

Since the first term on the right of (3.6) is independent of the disturbances we identify it with $2\pi U/\Gamma$. In the corresponding relation for (3.5) there appears the sum

$$\sum_m \frac{((n-m)b + c)}{2[a^2 + ((n-m)b + c)^2]}.$$

It will be assumed that this is summed symmetrically about $m = n$. After some rearranging of terms the linear system of differential equations for the disturbances can be placed in the form

$$(3.7)\qquad \dot{\zeta}_n = \sum_{m=-\infty}^{\infty} A_{n-m}\zeta_m, \qquad n = 0, \pm 1, \pm 2, \cdots,$$

where ζ_m is a vector with the four components x^1_m, y^1_m, x^2_m, y^2_m, and A_p is a four by four matrix with elements which can be described in terms of the following defined coefficients:

$$
(3.8)\qquad \begin{cases} b_p(r,q) = \dfrac{(p+q)^2 - r^2}{((p+q)^2 + r^2)^2}, & b_0(0,0) = 0, \\ c_p(r,q) = \dfrac{2r(p+q)}{((p+q)^2 + r^2)^2}, & B(r,q) = \sum_{p=-\infty}^{\infty} b_p(r,q), \\ C(r,q) = \sum_{p=-\infty}^{\infty} c_p(r,q), & r = a/b, \qquad q = c/b. \end{cases}
$$

Let δ_p be 1 or 0 according as $p = 0$ or $p \neq 0$. In terms of the coefficients (3.8), the elements of the matrix $A_p = (a_p^{ij})$ are calculated to be those shown in Table 1.

Since the series defining $B(r, q)$, $C(r, q)$ are absolutely convergent the matrices A_p may be regarded as Fourier constants of a matrix of continuous functions F on $0 \leqq \mu \leqq 1$. Therefore Theorem 1 may be now applied to solve the system (3.7). To this end define the functions $g(r, q) = g(\mu; r, q)$, $h(r, q) = h(\mu; r, q)$ by the relations

$$
(3.9)\qquad [g(r,q)]_p = b_p(r,q), \qquad [h(r,q)]_p = c_p(r,q).
$$

Then it is clear that

$$
(3.10)\qquad [\bar{g}(r,q)]_p = b_{-p}(r,q), \qquad [\bar{h}(r,q)]_p = c_{-p}(r,q),
$$

where the bar denotes the complex conjugate functions. In terms of the functions g and h, the matrix $F = (f^{ij})$, which determines the solution $\zeta(t)$ (cf. Theorem 1), is given as shown in Table 2.

We shall now investigate the stability of the linear system (3.7). It will be convenient to do this separately for the cases $q \neq \frac{1}{2}$, and $q = \frac{1}{2}$.

3.1. The Case $q \neq \frac{1}{2}$. THEOREM 3. *The linear system* (3.7) *where* $A_p = (a_p^{ij})$ *is given as in Table* 1, *is unstable* when* $q \neq \frac{1}{2}$.

PROOF. In order to show this, it is sufficient, by Theorem 2(a) to exhibit one value of μ for which at least one characteristic root of $F(\mu)$ has a positive real part when $q \neq \frac{1}{2}$. The characteristic roots of $F(\mu)$ are the roots of the fourth degree equation $\det(\lambda E_4 - F(\mu)) = 0$, E_4 being the four by four unit matrix, and this equation is readily calculated to be of the form

$$
(3.11)\qquad \begin{aligned} &[(8\pi b^2\lambda/\Gamma)^2 - ((g_0^2 + C) - (h\bar{h} + g\bar{g}))]^2 \\ &\qquad + (g\bar{h} - h\bar{g})^2 + 4(hg_0 - Cg)(\bar{h}g_0 - C\bar{g}) = 0. \end{aligned}
$$

In (3.11) g_0 is the real function defined by

$$
(3.12)\qquad g_0 = g_0(r,q) = B(r,q) - B(0,0) + g(0,0),
$$

and g, h are defined by (3.9). At $\mu = \frac{1}{2}$ *all* the functions g_0, g, h are real, and consequently (3.11) then takes the form

$$
(3.11')\qquad [(8\pi b^2\lambda/\Gamma)^2 - ((g_0^2 + C^2) - (h^2 + g^2))]^2 + 4(hg_0 - Cg)^2 = 0, \qquad \mu = \tfrac{1}{2}.
$$

* Unless stated otherwise "stable" and "unstable" are always meant with respect to all initial vector functions φ_0, $\| \varphi_0 \| < \infty$.

TABLE 1
$(8\pi b^3/\Gamma)a_p^{ij}$

i \ j	1	2	3	4
1	$-C(r, q)\delta_p$	$(B(r, q) - B(0, 0))\delta_p + b_p(0, 0)$	$c_p(r, q)$	$-b_p(r, q)$
2	$(B(r, q) - B(0, 0))\delta_p + b_p(0, 0)$	$C(r, q)\delta_p$	$-b_p(r, q)$	$-c_p(r, q)$
3	$-c_{-p}(r, q)$	$b_{-p}(r, q)$	$C(r, q)\delta_p$	$(B(0, 0) - B(r, q))\delta_p - b_p(0, 0)$
4	$b_{-p}(r, q)$	$c_{-p}(r, q)$	$(B(0, 0) - B(r, q))\delta_p - b_p(0, 0)$	$-C(r, q)\delta_p$

TABLE 2
$(8\pi b^3/\Gamma)f^{ij}$

i \ j	1	2	3	4
1	$-C(r, q)$	$B(r, q) - B(0, 0) + g(0, 0)$	$h(r, q)$	$-g(r, q)$
2	$B(r, q) - B(0, 0) + g(0, 0)$	$C(r, q)$	$-g(r, q)$	$-h(r, q)$
3	$-\bar{h}(r, q)$	$\bar{g}(r, q)$	$C(r, q)$	$B(0, 0) - B(r, q) - g(0, 0)$
4	$\bar{g}(r, q)$	$\bar{h}(r, q)$	$B(0, 0) - B(r, q) - g(0, 0)$	$-C(r, q)$

It is easily seen that the roots of (3.11′) must be of the form $\pm(\lambda_r \pm i\lambda_i)$, λ_r, λ_i real. From this fact, in order to prove instability, it is sufficient to show $\lambda_r \neq 0$ at $\mu = \frac{1}{2}$, and this will certainly be the case if $hg_0 - Cg \neq 0$ at $\mu = \frac{1}{2}$. We prove the latter inequality.

Assume $hg_0 = Cg$ at $\mu = \frac{1}{2}$, and that $C \neq 0$. (The latter assumption is equivalent to the fact that $q \neq 0, \frac{1}{2}$. See the explicit form of $C(r, q)$ in (3.18) below.) Then $g_0^2 - g^2 = g_0^2 - (h^2 g_0^2/C^2) = (g_0^2/C^2)(C^2 - h^2)$, and consequently $g_0^2 - g^2$ must have the same sign as $C^2 - h^2$, if $C \neq 0$. This will be shown not to be the case.

Now

$$g(\tfrac{1}{2}; r, q) = \sum_{p=-\infty}^{\infty} (-1)^p b_p(r, q) = \sum_{p=-\infty}^{\infty} b_{2p}(r, q) - \sum_{p=-\infty}^{\infty} b_{2p+1}(r, q), \tag{3.13}$$

and a similar relation holds for $h(\frac{1}{2}; r, q)$. It is easily verified that the coefficients $b_p(r, q)$ and $c_p(r, q)$ satisfy

$$4b_{2p}(r, q) = b_p(r/2, q/2), \qquad 4c_{2p}(r, q) = c_p(r/2, q/2). \tag{3.14}$$

Therefore

$$4 \sum_{p=-\infty}^{\infty} b_{2p}(r, q) = B(r/2, q/2), \qquad 4 \sum_{p=-\infty}^{\infty} c_{2p}(r, q) = C(r/2, q/2), \tag{3.15}$$

and since

$$B(r, q) = \sum_{p=-\infty}^{\infty} b_{2p}(r, q) + \sum_{p=-\infty}^{\infty} b_{2p+1}(r, q), \tag{3.16}$$

it follows from (3.13), (3.15), (3.16) that

$$\begin{cases} g(\frac{1}{2}; r, q) = \frac{1}{2}B(r/2, q/2) - B(r, q), \\ h(\frac{1}{2}; r, q) = \frac{1}{2}C(r/2, q/2) - C(r, q). \end{cases} \tag{3.17}$$

The explicit forms of $B(r, q)$ and $C(r, q)$ are given by (cf. [5])

$$\begin{cases} B(0, 0) = \pi^2/3, \qquad B(r, q) = 2\pi^2 \cdot \dfrac{1 - \cos 2\pi q \cosh 2\pi r}{(\cos 2\pi q - \cosh 2\pi r)^2}, \\ C(r, q) = 2\pi^2 \cdot \dfrac{\sin 2\pi q \sinh 2\pi r}{(\cos 2\pi q - \cosh 2\pi r)^2}, \qquad r \neq 0. \end{cases} \tag{3.18}$$

Therefore, from (3.17), (3.18), at $\mu = \frac{1}{2}$, $(C = C(r, q))$,

$$C^2 - h^2 = -\pi^4 \cdot \frac{\sin^2 \pi q \sinh^2 \pi r}{(\cos^2 \pi q - \cosh^2 \pi r)^2} < 0. \tag{3.19}$$

After much tedious reduction it can be shown that at $\mu = \frac{1}{2}$,

$$g_0^2 - g^2 = \frac{\pi^4}{4} \cdot \frac{(2 - (\cos^2 \pi q + \cosh^2 \pi r)^2)^2}{(\cos^2 \pi q - \cosh^2 \pi r)^2} > 0. \tag{3.20}$$

Therefore, comparing (3.19) and (3.20), if $q \neq 0, \frac{1}{2}$, $hg_0 \neq Cg$ at $\mu = \frac{1}{2}$, and this implies the instability of (3.7) in these cases.

For the case $q = 0$, we have $C(r, 0) = 0$, and hence $h(\frac{1}{2}; r, 0) = 0$ (cf. (3.17), (3.18)). Therefore the characteristic equation at $\mu = \frac{1}{2}$ (3.11′) reduces to

$$[(8\pi b^2\lambda/\Gamma)^2 - (g_0^2 - g^2)]^2 = 0, \tag{3.11″}$$

and since $g_0(\frac{1}{2}; r, 0) > g(\frac{1}{2}; r, 0)$* it follows that there exists a characteristic root $\lambda(\mu)$ such that $R\lambda(\frac{1}{2}) > 0$, and this is sufficient for instability by Theorem 2(a).

Remarks. The contradiction (3.19), (3.20) is patterned after Rosenhead [5]. He considered disturbances of a very particular nature, namely, $x_n^1 = \alpha^1(t) \cos(n + q)\omega$, $y_n^1 = \beta^1(t) \cos(n + q)\omega$, $x_n^2 = \alpha^2(t) \cos(n - q)\omega$, $y_n^2 = \beta^2(t) \cos(n - q)\omega$, where $0 \leqq \omega \leqq 2\pi$. By stability Rosenhead meant that the amplitudes $\alpha^1(t)$, $\beta^1(t)$, $\alpha^2(t)$, $\beta^2(t)$ should remain bounded as $t \to +\infty$. Under his assumption of equal amplitude for all vortices it is clear that $\| \zeta(0) \|^2 = \infty$, if $\omega = 0$, for example. However, his result was that, with respect to these disturbances, the only possible case of stability (in his sense) is the von Kármán case of $q = \frac{1}{2}$, $\cosh^2 \pi r = 2$. Actually the system in the latter case is not stable in his sense or ours. See the comments following Theorem 4.

3.2. The Case $q = \frac{1}{2}$. For this spacing $C(r, \frac{1}{2}) = 0$, $b_{p-1}(r, \frac{1}{2}) = b_{-p}(r, \frac{1}{2})$, and $c_{p-1}(r, \frac{1}{2}) = -c_{-p}(r, \frac{1}{2})$. Therefore

$$\bar{g}(r, \tfrac{1}{2}) = e^{2\pi i\mu} g(r, \tfrac{1}{2}), \qquad \bar{h}(r, \tfrac{1}{2}) = -e^{2\pi i\mu} h(r, \tfrac{1}{2}), \tag{3.21}$$

and hence $e^{\pi i\mu} g(r, \frac{1}{2})$ is real, $e^{\pi i\mu} h(r, \frac{1}{2})$ is pure imaginary, while g_0 is always real. These functions at $q = \frac{1}{2}$ are expressible in terms of elementary functions as follows (see (3.8), (3.9), and [3], p. 228)

$$\begin{cases} e^{\pi i\mu} g(\mu; r, \frac{1}{2}) = \dfrac{\pi^2 \cosh 2\pi r\mu}{\cosh^2 \pi r} - \dfrac{2\pi^2 \mu \cosh \pi r(1 - 2\mu)}{\cosh \pi r}, \\ g_0(\mu; r, \frac{1}{2}) = \dfrac{\pi^2}{\cosh^2 \pi r} - 2\pi^2 \mu(1 - \mu), \\ e^{\pi i\mu} h(\mu; r, \frac{1}{2}) = i\left\{ \dfrac{2\pi^2 \mu \sinh \pi r(1 - 2\mu)}{\cosh \pi r} + \dfrac{\pi^2 \sinh 2\pi r\mu}{\cosh^2 \pi r} \right\}. \end{cases} \tag{3.22}$$

The four roots of the characteristic equation (3.11) are then calculated to be

$$8\pi b^2 \lambda/\Gamma = \pm[e^{\pi i\mu} h \pm (g_0^2 - e^{2\pi i\mu} g^2)^{1/2}], \qquad q = \tfrac{1}{2}. \tag{3.23}$$

Therefore, in order that no root have a positive real part,

$$g_0^2 \leqq (e^{\pi i\mu} g)^2, \qquad q = \tfrac{1}{2}, \tag{3.24}$$

must hold for $0 \leqq \mu \leqq 1$. At $\mu = \frac{1}{2}$, $e^{\pi i\mu} g = 0$, and therefore a further necessary condition for stability is that $g_0(\frac{1}{2}; r, \frac{1}{2}) = 0$. From the definition of g_0 this condition becomes

$$\cosh^2 \pi r = 2. \tag{3.25}$$

* See the calculation in Lamb [3], pp. 227–228.

Lamb ([3], pp. 228, 229) has shown that when r satisfies (3.25), then $g_0^2 < (e^{\pi i\mu}g)^2$ for $0 < \mu < \frac{1}{2}$ and $\frac{1}{2} < \mu < 1$. For $\mu = 0$ and $\mu = 1$, $e^{\pi i\mu}h = 0$ and $g_0^2 = (e^{\pi i\mu}g)^2$, and at $\mu = \frac{1}{2}$, $e^{\pi i\mu}h = i\pi^2/2$. Thus, when (3.25) is true, all characteristic roots of (3.11) are pure imaginary for $0 < \mu < 1$, and zero for $\mu = 0$ and $\mu = 1$. Thus (3.25) implies (3.24). However (3.25) is not sufficient for stability, and in fact instability actually occurs.

In order to exhibit this we first calculate the solution explicitly, that is, the matrix e^{Ft}. For $q = \frac{1}{2}$ the matrix F has the form

$$\frac{8\pi b^2 F}{\Gamma} = \begin{pmatrix} 0 & g_0 & h & -g \\ g_0 & 0 & -g & -h \\ -\bar{h} & \bar{g} & 0 & -g_0 \\ \bar{g} & \bar{h} & -g_0 & 0 \end{pmatrix}. \tag{3.26}$$

We write $F = F_1 + F_2$, where

$$\frac{8\pi b^2 F_1}{\Gamma} = \begin{pmatrix} 0 & g_0 & 0 & -g \\ g_0 & 0 & -g & 0 \\ 0 & \bar{g} & 0 & -g_0 \\ \bar{g} & 0 & -g_0 & 0 \end{pmatrix}, \quad \frac{8\pi b^2 F_2}{\Gamma} = \begin{pmatrix} 0 & 0 & h & 0 \\ 0 & 0 & 0 & -h \\ -\bar{h} & 0 & 0 & 0 \\ 0 & \bar{h} & 0 & 0 \end{pmatrix}. \tag{3.27}$$

It is easy to see that F_1, F_2 commute, and hence

$$e^{Ft} = e^{F_1 t}e^{F_2 t}. \tag{3.28}$$

Also

$$\left(\frac{8\pi b^2}{\Gamma}\right)^2 F_1^2 = (g_0^2 - e^{2\pi i\mu}g^2)E_4, \qquad \left(\frac{8\pi b}{\Gamma}\right)^2 F_2^2 = e^{2\pi i\mu}h^2 E_4. \tag{3.29}$$

Therefore, by a direct substitution of (3.29) into the series for $e^{F_1 t}$ and $e^{F_2 t}$, we have, on putting

$$(8\pi b^2/\Gamma)\mathfrak{g} = (e^{2\pi i\mu}g^2 - g_0^2)^{1/2}, \qquad (8\pi b^2/\Gamma)i\mathfrak{h} = e^{\pi i\mu}h, \tag{3.30}$$

the following formulae

$$\begin{cases} e^{F_1 t} = (\cos \mathfrak{g}t)E_4 + (\sin \mathfrak{g}t/\mathfrak{g})F_1, \\ e^{F_2 t} = (\cos \mathfrak{h}t)E_4 + (\sin \mathfrak{h}t/\mathfrak{h})F_2, \end{cases} \tag{3.31}$$

where it is understood that all functions are evaluated at $q = \frac{1}{2}$ and $\cosh^2 \pi r = 2$.

From (3.22) and (3.26) it follows that $F(0) \neq 0$, and therefore $F(\mu)$ has non-simple elementary divisors associated with the characteristic root $\lambda = 0$ at $\mu = 0$. Therefore Theorem 2 cannot be applied in this case. However, we show directly that this behavior at $\mu = 0$ is enough for instability in the present case.

The definition of F_2 (3.27) implies $F_2^* = -F_2$, and hence $(e^{F_2 t})^* = e^{-F_2 t}$, that is, $e^{F_2 t}$ is a unitary matrix. Consequently if $\varphi_0 \in L^2(0, 1)$ we have $|\, e^{Ft}\varphi_0\,|^2 = |\, e^{F_1 t}\varphi_0\,|^2$. Let φ_0 be a vector with first component equal to $\tilde{\varphi}_0$ and all other components zero. Then from (3.31),

$$| e^{\mathcal{J}t}\varphi_0 |^2 = | \tilde{\varphi}_0 |^2[\cos^2 \mathfrak{g}t + (\sin^2 \mathfrak{g}t/\mathfrak{g}^2)(| g_0 |^2 + | g |^2)],$$

which is greater than or equal to $| \tilde{\varphi}_0 |^2 (\sin^2 \mathfrak{g}t/\mathfrak{g}^2)(| g_0 |^2 + | g |^2)$. Since $g_0(0) = g(0) = \pi^2/2$, and because $g_0(\mu)$ and $g(\mu)$ are continuous at $\mu = 0$, there exists a μ_1, $0 < \mu_1 < \frac{1}{2}$, such that

$$| g_0 |^2 + | g |^2 \geqq \pi^4/4, \qquad 0 \leqq \mu \leqq \mu_1 .$$

Then for fixed $t > 0$,

$$|| \zeta(t) ||^2 = || e^{\mathcal{J}t}\varphi_0 ||^2 \geqq (\pi^4/4) \int_0^{\mu_1} (\sin^2 \mathfrak{g}t/\mathfrak{g}^2) | \tilde{\varphi}_0 |^2 \, d\mu. \tag{3.32}$$

In order to appraise the integral in (3.32) we investigate the behavior of $\mathfrak{g}$ near $\mu = 0$. Let $\mathfrak{G} = \mathfrak{g}^2$. By direct calculation from (3.22) we have $\mathfrak{G}(0) = 0$, $(d\mathfrak{G}/d\mu)(0) = 0$, and $(d^2\mathfrak{G}/d\mu^2)(0) = 2\pi^4 \cdot (\pi^2 r^2 + 2\sqrt{2}\pi r - 2)(\Gamma/8\pi b^2)^2$ which is positive, for $\pi r = .8814 \cdots$. Thus $\mathfrak{G}(\mu)$ is increasing for μ sufficiently close to $\mu = 0$, say for $0 \leqq \mu \leqq \mu_2$. If $\mu_3 = \min(\mu_1, \mu_2)$, and t is chosen so large that $\pi^2/4t^2 < \mathfrak{G}(\mu_3)$, then there exists, by continuity, a δ, $0 < \delta < \mu_3$, such that $\mathfrak{G}(\delta) = \pi^2/4t^2$, and $0 \leqq \mathfrak{G}(\mu) < \pi^2/4t^2$ for $0 \leqq \mu < \delta$. Clearly $\delta = \delta(t) \to +0$, as $t \to +\infty$. Also, by Taylor's formula

$$\frac{\pi^2}{4t^2} = \mathfrak{G}(\delta) = \frac{d^2\mathfrak{G}}{d\mu^2}(0) \cdot \delta^2 + o(\delta^2), \qquad \delta \to +0,$$

or

$$\delta^2 t^2 = \frac{\pi^2}{4}\left(\frac{1}{2}\frac{d^2\mathfrak{G}}{d\mu^2}(0) + o(1)\right)^{-1}, \qquad t \to +\infty. \tag{3.33}$$

Now, returning to (3.32), we see that since $\mathfrak{g}(\delta)t = \pi/2$, and $(\sin \mathfrak{g}t/\mathfrak{g}t) > \frac{1}{2}$ for $0 \leqq \mathfrak{g}t \leqq \pi/2$, that is, for $0 \leqq \mu \leqq \delta$, then

$$|| \zeta(t) ||^2 \geqq (\pi^4 t^2/36) \int_0^{\delta} | \tilde{\varphi}_0 |^2 \, d\mu. \tag{3.34}$$

If $| \tilde{\varphi}_0 |^2 = \sigma\mu^\rho$, where σ is a positive constant and $-1 < \rho < 1$, then from (3.33) and (3.34) it follows that $|| \zeta(t) || \to +\infty$, $t \to +\infty$. This example proves that the system (3.7) is unstable for $q = \frac{1}{2}$ even though $\cosh^2 \pi r = 2$. Also, since σ can be made as small as desired, this example shows that instability occurs for perturbations $\zeta(t)$ whose initial magnitudes $|| \zeta(0) || = || \varphi_0 ||$ are arbitrarily small.

Unlike the cases where $q \neq \frac{1}{2}$, however, when $q = \frac{1}{2}$, $\cosh^2 \pi r = 2$, and the initial vector function φ_0 is restricted by certain symmetry considerations, then the system (3.7) (and Table 1) becomes stable with respect to this subclass of initial functions.

Definition. The sub-set S of initial vector functions $\varphi_0 = (\varphi_0^i) \in L^2(0, 1)$ is defined to be all those φ_0 for which there exists a $\hat{\mu}$, $0 < \hat{\mu} < \frac{1}{2}$, such that

$$\begin{cases} \varphi_0^3 = e^{\pi i\mu}\varphi_0^1, \qquad \varphi_0^4 = e^{\pi i\mu}\varphi_0^2, \qquad 0 \leqq \mu \leqq \hat{\mu}, \\ \varphi_0^3 = -e^{\pi i\mu}\varphi_0^1, \qquad \varphi_0^4 = -e^{\pi i\mu}\varphi_0^2. \qquad 1 - \hat{\mu} \leqq \mu \leqq 1. \end{cases} \tag{3.35}$$

We prove (3.7) is stable with respect to the set S if $q = \frac{1}{2}$ and $\cosh^2 \pi r = 2$. In fact we show that if $\varphi_0 \in S$, then $| e^{P_1 t}\varphi_0 |^2 \leqq M | \varphi_0 |^2$, $0 \leqq \mu \leqq 1$, where M is a constant independent of μ and t. As mentioned previously $e^{P_2 t}$ is unitary and hence it suffices to show that $| e^{P_1 t}\varphi_0 |^2 \leqq M | \varphi_0 |^2$. If $\varphi_0 \in S$ and $0 \leqq \mu \leqq \hat{\mu}$, then from (3.31)

$$e^{P_1 t}\varphi_0 = (\cos \mathfrak{g}t)\varphi_0 + (\hat{\mathfrak{g}} \sin \mathfrak{g}t)\hat{\varphi}_0 ,$$

where $\hat{\mathfrak{g}} = (e^{\pi i\mu}g - g_0)/8\pi b^2\mathfrak{g}$, and $\hat{\varphi}_0$ is the vector with components $-\varphi_0^2$, $-\varphi_0^1$, $e^{\pi i\mu}\varphi_0^2$, $e^{\pi i\mu}\varphi_0^1$. Clearly $| \hat{\varphi}_0 | = | \varphi_0 |$, and therefore $| e^{P_1 t}\varphi_0 | \leqq (1 + | \hat{\mathfrak{g}} |) | \varphi_0 |$ for $0 \leqq \mu \leqq \hat{\mu}$. From the definition of $\mathfrak{g}$ (cf. (3.30)) we have

$$\mathfrak{g} = (e^{\pi i\mu}g - g_0)^{1/2}/(e^{\pi i\mu}g + g_0)^{1/2}.$$

Since $g_0^2 < (e^{\pi i\mu}g)^2$, $0 < \mu < \frac{1}{2}$, and $e^{\pi i\mu}g = g_0 = \pi^2/2$ at $\mu = 0$, it follows that $e^{\pi i\mu}g + g_0 \neq 0$, $0 \leqq \mu \leqq \hat{\mu}$. Also $e^{\pi i\mu}g - g_0 \to 0$, $\mu \to +0$. Therefore $\hat{\mathfrak{g}}$ is bounded on $0 \leqq \mu \leqq \hat{\mu}$, and there exists a constant $M_1 > 0$, independent of μ and t, such that $| e^{P_1 t}\varphi_0 |^2 \leqq M_1 | \varphi_0 |^2$, for $\varphi_0 \in S$, and $0 \leqq \mu \leqq \hat{\mu}$. A similar argument shows that this is true on the interval $1 - \hat{\mu} \leqq \mu \leqq 1$, for a second constant $M_2 > 0$. Now consider the interval $\hat{\mu} \leqq \mu \leqq 1 - \hat{\mu}$. Here the components of φ_0 are not restricted other than that $| \varphi_0 |^2$ must be integrable. By a direct calculation from (3.31) we have

$$e^{P_1 t}\varphi_0 = (\cos \mathfrak{g}t)\varphi_0 + (\Gamma \sin \mathfrak{g}t/8\pi b^2\mathfrak{g})\chi_0 ,$$

where χ_0 is the vector with components $\varphi_0^2 g_0 - \varphi_0^4 g$, $\varphi_0^1 g_0 - \varphi_0^3 g$, $\varphi_0^3 g - \varphi_0^4 g_0$, $\varphi_0^1 g - \varphi_0^3 g_0$. Therefore $| \chi_0 | \leqq (| g_0 | + | g |) | \varphi_0 |$, and hence

$$| e^{P_1 t}\varphi_0 | \leqq (1 + \Gamma(| g_0 | + | g |)/8\pi b^2\mathfrak{g}) | \varphi_0 |.$$

On the interval $\hat{\mu} \leqq \mu \leqq 1 - \hat{\mu}$, $(8\pi b^2/\Gamma)\mathfrak{g} = (e^{2\pi i\mu}g^2 - g_0^2)^{1/2}$ vanishes only at $\mu = \frac{1}{2}$, and both $e^{\pi i\mu}g$ and g_0 vanish there. However, it is not difficult to see from (3.22), when $\cosh^2 \pi r = 2$, that $g_0/e^{\pi i\mu}g \to 0$, $\mu \to \frac{1}{2}$. Thus $\Gamma | g_0 |/8\pi b^2\mathfrak{g} \to 0$, and $\Gamma | g |/8\pi b^2\mathfrak{g} \to 1$, as $\mu \to \frac{1}{2}$. Consequently $1 + \Gamma(| g_0 | + | g |)/8\pi b^2\mathfrak{g}$ is bounded on $\hat{\mu} \leqq \mu \leqq 1 - \hat{\mu}$, and the existence of a constant $M > 0$ such that $| e^{P_1 t}\varphi_0 |^2 \leqq M | \varphi_0 |^2$, $0 \leqq \mu \leqq 1$, is proved.

We summarize our results for the case $q = \frac{1}{2}$ in the following theorem.

THEOREM 4. (a) *If $q = \frac{1}{2}$, a necessary condition that the linear system* (3.7), *where A_p is given as in Table* 1, *be stable with respect to all initial vector functions $\varphi_0 \in L^2(0, 1)$ is that* $\cosh^2 \pi r = 2$.

(b) *Corresponding to any such φ_0 (if $q = \frac{1}{2}$ and* $\cosh^2 \pi r = 2$) *the solution $\zeta = (\zeta_n)$ of* (3.7) *is given by*

$$\zeta_n(t) = [e^{Pt}\varphi_0]_n , \qquad n = 0, \pm 1, \pm 2, \cdots ,$$

where e^{Pt} is determined by (3.27), (3.28), (3.30), (3.31).

(c) *The condition* $\cosh^2 \pi r = 2$ *is not sufficient for stability when $q = \frac{1}{2}$, for there exist initial functions φ_0 for which $\| \varphi_0 \|$ is arbitrarily small, and such that $\| \zeta(t) \| \to +\infty$, $t \to +\infty$.*

(d) *If* $q = \frac{1}{2}$, $\cosh^2 \pi r = 2$, *then the linear system* (3.7) (*and Table* 1) *is stable with respect to the set* S *of initial vector functions* (*cf.* (3.35)).

Clearly S is not the largest set of $\varphi_0 \in L^2(0, 1)$ for which $\limsup \| \zeta(t) \| < \infty$, $t \to +\infty$. The characterization of the set of all $\varphi_0 \in L^2(0, 1)$ for which (3.7) (and Table 1) is stable when $q = \frac{1}{2}$, $\cosh^2 \pi r = 2$, remains an open question.

3.3. Discussion of von Kármán's result. It should be pointed out how Theorem 4 compares with the result von Kármán obtained [1], [2]. He considered only the cases $q = 0, \frac{1}{2}$, and assumed that the displacements were of the form

$$x_n^1 = \alpha^1(t)e^{in\omega}, \quad y_n^1 = \beta^1(t)e^{in\omega}, \quad x_n^2 = \alpha^2(t)e^{in\omega}, \quad y_n^2 = \beta^2(t)e^{in\omega}, \tag{3.36}$$

where $0 \leqq \omega \leqq 2\pi$. This assumption reduces the system (3.7) (with Table 1) to a system of *four* first order linear differential equations for α^1, β^1, α^2, β^2. He defined this finite system to be stable if each of the amplitudes α^1, β^1, α^2, β^2 is bounded as $t \to +\infty$, for all ω, $0 \leqq \omega \leqq 2\pi$. Then assuming that all solutions were proportional to $e^{\lambda t}$ he concluded that when $q = 0$ the system is unstable for all values of r, and when $q = \frac{1}{2}$, stability occurs if, and only if, $\cosh^2 \pi r = 2$. The latter ("if") conclusion is unwarranted due to the fact that it is not true that all solutions are proportional to $e^{\lambda t}$ for *all* ω, $0 \leqq \omega \leqq 2\pi$. For $q = \frac{1}{2}$ the assumption (3.36) reduces (3.7) to the following system

$$\dot{u} = \bar{F}(\omega/2\pi)u, \tag{3.37}$$

where u is the vector with components α^1, β^1, α^2, β^2. A fundamental matrix for (3.37) is given by

$$e^{\bar{F}(\omega/2\pi)t}. \tag{3.38}$$

At $\omega = 0$, $F_2 = 0$, $\bar{F}_1 = F_1 \neq 0$, and in fact, since $g_0(0) = g(0) = \pi^2/2$,

$$(16b^2/\Gamma\pi)F_1(0) = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}. \tag{3.39}$$

From (3.31) it follows that

$$e^{\bar{F}(0)t} = E_4 + tF_1(0), \tag{3.40}$$

and since the columns of (3.38) form four linearly independent solution vectors of (3.37), it is obvious from (3.39), (3.40) that the system (3.37) is unstable when $\omega = 0$.

On the other hand if (3.36) is assumed, then $\| \zeta(t) \| = +\infty$ for *all* $t \geqq 0$, unless $\zeta(0) = 0$, and in view of energy considerations this is a further objection to the von Kármán procedure.

Of course, the stability of the motion of the vortex street itself is governed by the stability of the solutions of the infinite *non-linear* system (cf. (3.3), (3.5))

for which (3.7) is only the linear approximation. It remains to determine the connection between the stability of (3.7) and that of the non-linear system. We expect to return to this later.

4. Double Vortex Streets. We now consider a pair of von Kármán vortex streets, each having the spacing $q = \frac{1}{2}$, and confine our attention to the two configurations illustrated in Figures 2 and 3. For the double vortex street of Type A there are four rows of vortices located in the (x, y)-plane as follows:

Type A Double Vortex Street

Row	*Vortex Strength*	*Vortex Distribution*
1	Γ_1	$((2n-1)b,\ 2a+d)$
2	$-\Gamma_2$	$(2nb,\ d)$
3	Γ_2	$((2n-1)b,\ -d)$
4	$-\Gamma_1$	$(2nb,\ -(2a+d))$

$$n = 0, \pm 1, \pm 2, \cdots .$$

The double street of Type B is described by the following table.

Type B Double Vortex Street

Row	*Vortex Strength*	*Vortex Distribution*
1	Γ_1	$(2nb,\ 2a+d)$
2	$-\Gamma_2$	$((2n-1)b,\ d)$
3	Γ_2	$((2n-1)b,\ -d)$
4	$-\Gamma_1$	$(2nb,\ -(2a+d))$

$$n = 0, \pm 1, \pm 2, \cdots .$$

The principal reason for considering these two types is that both patterns have actually been observed in the wake of adjacent cylinders (see the experimental reports [4], [7]).

Let the n-th vortex in the i-th row be displaced by the amount (x_n^i, y_n^i), $i = 1, \cdots, 4$. Let $\Delta_{mn}^{ij}(x)$ denote the x-distance from the m-th displaced vortex in the i-th row to the n-th displaced vortex in the j-th row. Similarly define $\Delta_{mn}^{ij}(y)$ to be the corresponding y-distance. Let r_{mn}^{ij} be the distance defined by $(r_{mn}^{ij})^2 = (\Delta_{mn}^{ij}(x))^2 + (\Delta_{mn}^{ij}(y))^2$. For a Type B vortex street, for example,

$$\Delta_{mn}^{21}(x) = 2b(n - m + \tfrac{1}{2}) + (x_n^1 - x_m^2), \qquad \Delta_{mn}^{21}(y) = 2a + (y_n^1 - y_m^2).$$

The x- and y-components of velocity of the n-th vortex in the j-th row are given by

$$(4.1)\quad \begin{cases} 2\pi(\dot{x}_n^j + U^j) = \sum_m \left[-\Gamma_1 \dfrac{\Delta_{mn}^{1j}(y)}{(r_{mn}^{1j})^2} + \Gamma_2 \dfrac{\Delta_{mn}^{2j}(y)}{(r_{mn}^{2j})^2} - \Gamma_2 \dfrac{\Delta_{mn}^{3j}(y)}{(r_{mn}^{3j})^2} + \Gamma_1 \dfrac{\Delta_{mn}^{4j}(y)}{(r_{mn}^{4j})^2} \right], \\ 2\pi\dot{y}_n^j = \sum_m \left[\Gamma_1 \dfrac{\Delta_{mn}^{1j}(x)}{(r_{mn}^{1j})^2} - \Gamma_2 \dfrac{\Delta_{mn}^{2j}(x)}{(r_{mn}^{2j})^2} + \Gamma_2 \dfrac{\Delta_{mn}^{3j}(x)}{(r_{mn}^{3j})^2} - \Gamma_1 \dfrac{\Delta_{mn}^{4j}(x)}{(r_{mn}^{4j})^2} \right]. \end{cases}$$

Here U^j is the translational velocity of the j-th row in the x-direction, summations are from $m = -\infty$ to $m = +\infty$, excluding $m = n$ when a term contains $\Delta^{ij}_{mn}(x)$ or $\Delta^{ij}_{mn}(y)$.

We consider the infinite linear system resulting from (4.1) if terms of second and higher orders in the displacements x^i_n, y^i_n are neglected. If ζ_n denotes the vector with the eight components $x^1_n, y^1_n, \cdots, x^4_n, y^4_n$, then this infinite linear system can be placed in the form

$$\dot{\zeta}_n = \sum_{m=-\infty}^{\infty} A_{n-m}\zeta_m, \qquad n = 0, \pm 1, \pm 2, \cdots, \tag{4.2}$$

where A_p is a matrix with eight rows and columns.

In the transition from (4.1) to (4.2) certain sums of the general nature $\sum_{m\neq 0}(1/m)$ are encountered, and these are assumed to be summed symmetrically with respect to $m = 0$. Also the velocities U^j are identified with the terms which are independent of the disturbances x^i_n, y^i_n.

We shall determine the explicit form of the matrices A_p, or what is the same thing, the matrix $F = F(\mu)$ which determines the solution of (4.2). We then find *necessary* conditions for the stability of the system (4.2). It will be convenient to consider the two cases separately. The following ratios will be constantly used:

$$r = a/b, \qquad s = d/b, \qquad \gamma = \Gamma_1/\Gamma_2. \tag{4.3}$$

4.1. Type A Double Vortex Street. In terms of the parameters introduced in (4.3) the velocities U^i are given by

$$\begin{cases} 4\pi b U^1 = \sum_{m=-\infty}^{\infty}\left[\dfrac{\Gamma_2 r}{(m-\frac{1}{2})^2+r^2} - \dfrac{\Gamma_2(r+s)}{m^2+(r+s)^2} + \dfrac{\Gamma_1(2r+s)}{(m-\frac{1}{2})^2+(2r+s)^2}\right], \\ 4\pi b U^2 = \sum_{m=-\infty}^{\infty}\left[\dfrac{\Gamma_1 r}{(m+\frac{1}{2})^2+r^2} - \dfrac{\Gamma_2 s}{(m+\frac{1}{2})^2+s^2} + \dfrac{\Gamma_1(r+s)}{m^2+(r+s)^2}\right], \\ U^3 = U^2, U^4, = U^1. \end{cases} \tag{4.4}$$

Using the well-known expansions for $\tanh \pi w$ and $\coth \pi w$, the formulae in (4.4) may be written as

$$\begin{cases} 4bU^1 = \Gamma_2 \tanh \pi r - \Gamma_2 \coth \pi(r+s) + \Gamma_1 \tanh \pi(2r+s), \\ 4bU^2 = \Gamma_1 \tanh \pi r - \Gamma_2 \tanh \pi s + \Gamma_1 \coth \pi(r+s), \; U^3 = U^2, \; U^4 = U^1. \end{cases} \tag{4.5}$$

In order that the vortex system preserve its form independent of t we must have $U^1 = U^2$, and this results in the following condition for the ratio $\gamma = \Gamma_1/\Gamma_2$,

$$\gamma = \frac{\tanh \pi r - \coth \pi(r+s) + \tanh \pi s}{\tanh \pi r + \coth \pi(r+s) - \tanh \pi(2r+s)}. \tag{4.6}$$

Note that negative γ are not ruled out by (4.6).

We turn now to the description of the matrix F. All the elements of A_p can be described in terms of the following coefficients:

$$(4.7)\qquad \begin{cases} b_p(w) = \dfrac{(p+\frac{1}{2})^2 - w^2}{((p+\frac{1}{2})^2 + w^2)^2}, \qquad c_p(w) = \dfrac{2w(p+\frac{1}{2})}{((p+\frac{1}{2})^2+w^2)^2}. \\ B_l(w) = \sum_{p=-\infty}^{\infty} b_{p-l}(w). \end{cases}$$

We define functions g_i, h_i on $0 \leqq \mu \leqq 1$ as follows:

$$(4.8)\qquad \begin{cases} [g_1(w)]_p = b_p(w), \qquad [g_2(w)]_p = b_{p-1/2}(w), \\ [h_1(w)]_p = c_p(w), \qquad [h_2(w)]_p = c_{p-1/2}(w), \\ [g_4(v,w,\gamma)]_p = \gamma(b_{p-1/2}(0) + B_0(2v+w)\delta_p - B_{1/2}(0)\delta_p) \\ \qquad\qquad + (B_0(v) - B_{1/2}(v+w))\delta_p, \\ [g_5(v,w,\gamma)]_p = \gamma(B_{1/2}(v+w) - B_0(v))\delta_p - b_{p-1/2}(0) \\ \qquad\qquad + (B_{1/2}(0) - B_0(w))\delta_p. \end{cases}$$

Whenever a coefficient is not defined at $p = 0$ (for example, $b_{p-1/2}(0)$) it is understood that the corresponding function has 0 as its 0-th Fourier coefficient. In terms of elementary functions it is easy to show that (cf. [3], pp. 227–228)

$$(4.9)\qquad \begin{cases} e^{\pi i\mu} g_1(\mu; w) = \dfrac{\pi^2 \cosh 2\pi w\mu}{\cosh^2 \pi w} - \dfrac{2\pi^2\mu \cosh \pi w(1-2\mu)}{\cosh \pi w}, \\ g_2(\mu; w) = \dfrac{-\pi^2 \cosh 2\pi w\mu}{\sinh^2 \pi w} - \dfrac{2\pi^2\mu \sinh \pi w(1-2\mu)}{\sinh \pi w}, \\ -ie^{\pi i\mu} h_1(\mu; w) = \dfrac{2\pi^2\mu \sinh \pi w(1-2\mu)}{\cosh \pi w} + \dfrac{\pi^2 \sinh 2\pi w\mu}{\cosh^2 \pi w}, \\ -ih_2(\mu; w) = \dfrac{2\pi^2\mu \cosh \pi w(1-2\mu)}{\sinh \pi w} - \dfrac{\pi^2 \sinh 2\pi w\mu}{\sinh^2 \pi w}, \\ g_4(\mu; v, w, \gamma) = \gamma\left[2\pi^2\mu(\mu-1) + \dfrac{\pi^2}{\cosh^2 \pi(2v+w)}\right] \\ \qquad\qquad + \dfrac{\pi^2}{\cosh^2 \pi v} + \dfrac{\pi^2}{\sinh^2 \pi(v+w)}, \\ g_5(\mu; v, w, \gamma) = -\gamma\left[\dfrac{\pi^2}{\cosh^2 \pi v} + \dfrac{\pi^2}{\sinh^2 \pi(v+w)}\right] \\ \qquad\qquad - 2\pi^2\mu(\mu-1) - \dfrac{\pi^2}{\cosh^2 \pi w}. \end{cases}$$

In terms of the functions in (4.9) the elements of the matrix $F = (f^{ij})$ turn out to be given as shown in Table 3.

The stability properties of $\| \zeta \|$ are not affected by replacing F by $\bar{F} = U^{-1}FU$,

TABLE 3

$(8\pi b^2/\Gamma_2)f^{ij}$

Type A Double Vortex Street

i \ j	1	2	3	4	5	6	7	8
1	0	$g_4(r, s, \gamma)$	$-\overline{h}_1(r)$	$-\bar{g}_1(r)$	$-h_2(r+s)$	$g_2(r+s)$	$-\gamma\overline{h}_1(2r+s)$	$-\gamma\bar{g}_1(2r+s)$
2	$g_4(r, s, \gamma)$	0	$-\bar{g}_1(r)$	$\overline{h}_1(r)$	$g_2(r+s)$	$h_2(r+s)$	$-\gamma\bar{g}_1(2r+s)$	$\gamma\overline{h}_1(2r+s)$
3	$\gamma h_1(r)$	$\gamma g_1(r)$	0	$g_3(r, s, \gamma)$	$-h_1(s)$	$g_1(s)$	$\gamma h_2(r+s)$	$-\gamma g_2(r+s)$
4	$\gamma g_1(r)$	$-\gamma h_1(r)$	$g_3(r, s, \gamma)$	0	$g_1(s)$	$h_1(s)$	$-\gamma g_2(r+s)$	$-\gamma h_2(r+s)$
5	$\gamma h_2(r+s)$	$\gamma g_2(r+s)$	$\overline{h}_1(s)$	$-\bar{g}_1(s)$	0	$-g_4(r, s, \gamma)$	$-\gamma\overline{h}_1(r)$	$-\gamma\bar{g}_1(r)$
6	$\gamma g_2(r+s)$	$-\gamma h_2(r+s)$	$-\bar{g}_1(s)$	$-\overline{h}_1(s)$	$-g_4(r, s, \gamma)$	0	$-\gamma\bar{g}_1(r)$	$\gamma\overline{h}_1(r)$
7	$\gamma h_1(2r+s)$	$\gamma g_1(2r+s)$	$-h_2(r+s)$	$-g_2(r+s)$	$h_1(r)$	$g_1(r)$	0	$-g_4(r, s, \gamma)$
8	$\gamma g_1(2r+s)$	$-\gamma h_1(2r+s)$	$-g_2(r+s)$	$h_2(r+s)$	$g_1(r)$	$-h_1(r)$	$-g_4(r, s, \gamma)$	0

TABLE 4

Type A Double Vortex Street

$(8\pi b^2/\Gamma_2)F_1$

$$\begin{array}{llll}
\gamma e^{\gamma i s} h_1(2r+s) & g_4(r,s,\gamma) + \gamma e^{\gamma i s} g_1(2r+s) & e^{\gamma i s} h_1(r) - h_2(r+s) & -e^{\gamma i s} g_1(r) - g_2(r+s) \\
g_4(r,s,\gamma) - \gamma e^{\gamma i s} g_1(2r+s) & \gamma e^{\gamma i s} h_1(2r+s) & -e^{\gamma i s} g_1(r) + g_2(r+s) & -e^{\gamma i s} h_1(r) - h_2(r+s) \\
\gamma e^{\gamma i s} h_1(r) + \gamma h_2(r+s) & \gamma e^{\gamma i s} g_1(r) + \gamma g_2(r+s) & -e^{\gamma i s} h_1(s) & g_3(r,s,\gamma) - e^{\gamma i s} g_1(s) \\
\gamma e^{\gamma i s} g_1(r) - \gamma g_2(r+s) & -\gamma e^{\gamma i s} h_1(r) + \gamma h_2(r+s) & g_4(r,s,\gamma) + e^{\gamma i s} g_1(s) & -e^{\gamma i s} h_1(s),
\end{array}$$

$(8\pi b^2/\Gamma_2)F_2$

$$\begin{array}{llll}
-\gamma e^{\gamma i s} h_1(2r+s) & g_4(r,s,\gamma) - \gamma e^{\gamma i s} g_1(2r+s) & e^{\gamma i s} h_1(r) + h_2(r+s) & -e^{\gamma i s} g_1(r) + g_2(r+s) \\
g_4(r,s,\gamma) + \gamma e^{\gamma i s} g_1(2r+s) & -\gamma e^{\gamma i s} h_1(2r+s) & -e^{\gamma i s} g_1(r) - g_2(r+s) & -e^{\gamma i s} h_1(r) + h_2(r+s) \\
\gamma e^{\gamma i s} h_1(r) - \gamma h_2(r+s) & \gamma e^{\gamma i s} g_1(r) - \gamma g_2(r+s) & e^{\gamma i s} h_1(s) & g_3(r,s,\gamma) + e^{\gamma i s} g_1(s) \\
\gamma e^{\gamma i s} g_1(r) + \gamma g_2(r+s) & -\gamma e^{\gamma i s} h_1(r) - \gamma h_2(r+s) & g_4(r,s,\gamma) - e^{\gamma i s} g_1(s) & e^{\gamma i s} h_1(s).
\end{array}$$

where U is a unitary matrix. We make use of this remark in order to simplify matters. Let the unitary matrix U be given by

$$\sqrt{2}U = \begin{pmatrix} 1 & 0 & 0 & 0 & e^{\pi i\mu} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & e^{\pi i\mu} & 0 & 0 \\ 0 & 0 & e^{-\pi i\mu} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-\pi i\mu} & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -e^{\pi i\mu} & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & e^{\pi i\mu} \\ e^{-\pi i\mu} & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -e^{-\pi i\mu} & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then a calculation shows that $\tilde{F}$ assumes the simple form

$$\tilde{F} = \begin{pmatrix} F_1 & 0_4 \\ 0_4 & F_2 \end{pmatrix}$$

where 0_4 = four by four zero matrix, and F_1, F_2 are given by Table 4. Since $\det(\lambda E_8 - \tilde{F}) = \det(\lambda E_4 - F_1)\cdot\det(\lambda E_4 - F_2)$, it is sufficient to consider the characteristic roots of F_1 and F_2 separately.

We investigate the characteristic roots of F_1 at $\mu = \frac{1}{2}$. As can be seen from (4.9), $g_1(\frac{1}{2}; w) = 0$, $h_2(\frac{1}{2}; w) = 0$. If S is the unitary matrix

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix},$$

then $(8\pi b^2/\Gamma_2)\, S^{-1}F_1(1/2)\, S = G_1$, where

$$G_1 = \begin{pmatrix} a_1 & 0 & 0 & b_1 \\ 0 & a_2 & b_2 & 0 \\ 0 & \gamma b_2 & a_3 & 0 \\ \gamma b_1 & 0 & 0 & a_4 \end{pmatrix},$$

and

$$(4.10)\quad \begin{cases} a_1 = g_4(r, s, \gamma) + i\gamma h_1(2r + s),\ a_2 = -g_4(r, s, \gamma) + i\gamma h_1(2r + s), \\ a_3 = g_5(r, s, \gamma) - ih_1(s),\ a_4 = -g_5(r, s, \gamma) - ih_1(s), \\ b_1 = ih_1(r) + g_2(r + s),\ b_2 = ih_1(r) - g_2(r + s), \end{cases}$$

and these functions are evaluated at $\mu = \frac{1}{2}$. At this point the characteristic roots of F_1 are $(\Gamma_2/8\pi b^2)$ times those of G_1, and hence it suffices to discuss the latter. The characteristic equation for G_1 is given by $\det(\nu E_4 - G_1) = \Phi_1(\nu)\Phi_2(\nu) = 0$, where

$$\Phi_1(\nu) = \nu^2 - (a_1 + a_4)\nu + (a_1a_4 - \gamma b_1^2),$$

$$\Phi_2(\nu) = \nu^2 - (a_2 + a_3)\nu + (a_2a_3 - \gamma b_1^2).$$

From (4.10) it follows that $a_2 = -\bar{a}_1$, $a_3 = -\bar{a}_4$, $b_2 = -\bar{b}_1$, and therefore $\Phi_2(\nu) = \bar{\Phi}_1(-\bar{\nu})$. Hence, if ν is a characteristic root of G_1, $-\bar{\nu}$ is also. Thus a necessary condition for stability is that $R\nu = 0$. If ν_1, ν_2 are the roots of $\Phi_1(\nu) = 0$, then clearly $\nu_1 + \nu_2 = a_1 + a_4$, $\nu_1\nu_2 = a_1a_4 - \gamma b_1^2$, and for stability we must have $a_1 + a_4$ pure imaginary and $a_1a_4 - \gamma b_1^2$ real and non-positive. But

$$a_1 + a_4 = (g_4(r, s, \gamma) - g_5(r, s, \gamma)) + i(\gamma h_1(2r + s) - h_1(s)),$$

and at $\mu = \frac{1}{2}$ it is easy to see from (4.9) that g_4, g_5, h_1 are real. Thus a necessary condition for stability is that $g_4(r, s, \gamma) = g_5(r, s, \gamma)$ at $\mu = \frac{1}{2}$.

If $G_2 = (8\pi b^2/\Gamma_2)S^{-1}F_2S$, then the characteristic equation of G_2 at $\mu = \frac{1}{2}$ is $\det(\nu E_4 - G_2) = \Phi_3(\nu)\Phi_4(\nu) = 0$, where

$$\Phi_3(\nu) = \nu^2 + (a_1 + a_4)\nu + (a_1a_4 - \gamma b_1^2),$$

$$\Phi_4(\nu) = \nu^2 + (a_2 + a_3)\nu + (a_2a_3 - \gamma b_2^2).$$

Therefore, $\Phi_3(\nu) = \bar{\Phi}_2(\bar{\nu})$, $\Phi_4(\nu) = \bar{\Phi}_1(\bar{\nu})$, and hence at $\mu = \frac{1}{2}$ we obtain no new information from G_2.

We summarize our results for the double vortex street of Type A in the following theorem.

THEOREM 5. (a) *The double vortex street of Type A preserves its form, independent of the time, if and only if*

$$\gamma = \frac{\tanh \pi r - \coth \pi(r + s) + \tanh \pi s}{\tanh \pi r + \coth \pi(r + s) - \tanh \pi(2r + s)}.$$

(b) *Corresponding to any initial* ζ^0, $\|\zeta^0\| < \infty$, *the solution of the infinite linear system* (4.2), *in the case of the Type A street, is given as in Theorem* 1 *where* F *is the matrix given in Table* 3.

(c) *Necessary for stability of the system* (4.2), *in the Type A case, is that* $g_4(\frac{1}{2}; r, s, \gamma) = g_5(\frac{1}{2}; r, s, \gamma)$ (*cf.* (4.9).)) *and that* $a_1a_4 - \gamma b_1^2$ *be real and non-positive, where* a_1, a_4, b_1 *are given by* (4.9), (4.10).

Remarks. The ratio $\gamma \to 1$ as $s \to +\infty$, and the condition $g_4(\frac{1}{2}; r, s, \gamma) = g_5(\frac{1}{2}; r, s, \gamma)$ becomes in the limit, as $s \to +\infty$, the von Kármán relation $\cosh^2 \pi r = 2$. As was seen in Theorem 4, this condition is not sufficient for stability in the single street case, and therefore it would be surprising if $g_4 = g_5$

at $\mu = \frac{1}{2}$ was sufficient for stability for the double vortex street of Type A. The conditions cited in Theorem 5 (a), (c) limit the possible pairs of values (r, s), and in order to ascertain whether stability actually occurs with these allowable values (even for a subclass of initial perturbations) one must first check to see that every characteristic root of F has a real part which is non-positive at every μ, $0 \leqq \mu \leqq 1$. In view of the complexity of F it appears that this is a job for some computing device, and so we do not pursue this situation further.

4.2. Type B Double Vortex Street. The method of dealing with this case is parallel to that used for the Type A double vortex street.

The velocities U^j are given by

$$(4.11)\quad \begin{cases} 4bU^1 = \Gamma_2 \tanh \pi r - \Gamma_2 \tanh \pi(r+s) + \Gamma_1 \coth \pi(2r+s), \\ 4bU^2 = \Gamma_2 \tanh \pi r - \Gamma_2 \coth \pi s + \Gamma_1 \tanh \pi(r+s), \\ U^3 = U^2, \qquad U^4 = U^1, \end{cases}$$

and from this it follows that the vortex configuration keeps its form independent of t if and only if

$$(4.12)\quad \gamma^{-1} = \frac{\tanh \pi r + \tanh \pi(r+s) - \coth \pi(2r+s)}{\tanh \pi r - \tanh \pi(r+s) + \coth \pi s}.$$

We note that γ^{-1} can be 0 for non-zero r and s.

In order to describe the matrix F in this case it is necessary to introduce two new functions defined by

$$(4.13)\quad \begin{cases} [g_6(v,w,\gamma)]_p = b_{p-1/2}(0) - B_{1/2}(0)\delta_p \\ \qquad\qquad + B_{1/2}(2v+w)\delta_p + \gamma^{-1}(B_0(v) - B_0(v+w))\delta_p, \\ [g_7(v,w,\gamma)]_p = B_0(v+w)\delta_p - B_0(v)\delta_p \\ \qquad\qquad + \gamma^{-1}(B_{1/2}(0)\delta_p - B_{1/2}(w)\delta_p - b_{p-1/2}(0)). \end{cases}$$

An easy calculation shows that g_6, g_7 may be expressed in terms of the hyperbolic functions as follows:

$$(4.14)\quad \begin{cases} g_6(\mu; v, w, \gamma) = 2\pi^2\mu(\mu-1) - \dfrac{\pi^2}{\sinh^2 \pi(2v+w)} \\ \qquad\qquad + \gamma^{-1}\left[\dfrac{\pi^2}{\cosh^2 \pi v} - \dfrac{\pi^2}{\cosh^2 \pi(v+w)}\right], \\ g_7(\mu; v, w, \gamma) = \dfrac{\pi^2}{\cosh^2 \pi(v+w)} - \dfrac{\pi^2}{\cosh^2 \pi v} \\ \qquad\qquad + \gamma^{-1}\left[\dfrac{\pi^2}{\sinh^2 \pi w} - 2\pi^2\mu(\mu-1)\right]. \end{cases}$$

The elements of $F = (f^{ij})$ may now be expressed in terms of the functions (4.9), (4.14), and this is shown in Table 5. As before, the investigation of the charac-

teristic roots of F may be simplified by considering the similar matrix $\tilde{F} = V^{-1}FV$, where V is the unitary matrix defined by

$$\sqrt{2}\,V = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then

$$\tilde{F} = \begin{pmatrix} F_1 & 0_4 \\ 0_4 & F_2 \end{pmatrix},$$

where F_1, F_2 are the matrices given in Table 6.

At $\mu = \frac{1}{2}$ the characteristic equation of $(8\pi b^2/\Gamma_1)F_1$ is the same as that for $(8\pi b^2/\Gamma_1)F_2$, and this is given by

$$\lambda^4 + \alpha\lambda^2 + \beta = 0, \tag{4.15}$$

where

$$\begin{cases} \alpha = 2\gamma^{-1}\beta_1\beta_2 - \alpha_1\alpha_2 - \alpha_3\alpha_4, \\ \beta = \alpha_1\alpha_2\alpha_3\alpha_4 - \gamma^{-1}(\alpha_1\alpha_2\beta_1^2 + \alpha_3\alpha_4\beta_2^2) + \gamma^{-2}\beta_1^2\beta_2^2, \end{cases} \tag{4.16}$$

and

$$\begin{cases} \alpha_1 = g_6(r, s, \gamma) + g_2(2r + s), & \alpha_2 = g_6(r, s, \gamma) - g_2(2r + s), \\ \alpha_3 = g_7(r, s, \gamma) + \gamma^{-1}g_2(s), & \alpha_4 = g_7(r, s, \gamma) - \gamma^{-1}g_2(s), \\ \beta_1 = h_1(r) + h_1(r + s), & \beta_2 = h_1(r) - h_1(r + s). \end{cases} \tag{4.17}$$

It is understood that the functions in (4.17) are evaluated at $\mu = \frac{1}{2}$, where they are all real. From (4.15) it follows that if λ is a root then so are $-\lambda$ and $\bar{\lambda}$. Hence, for stability, λ must be pure imaginary, that is λ^2 must be real and non-positive. But $2\lambda^2 = -\alpha \pm (\alpha^2 - 4\beta)^{1/2}$, and therefore we must have $\alpha^2 - 4\beta \geqq 0$, $\alpha \geqq 0$, and $\alpha \geqq (\alpha^2 - 4\beta)^{1/2}$, or $\beta \geqq 0$.

We summarize our discussion on the Type B vortex street in the following theorem.

THEOREM 6. (a) *The double vortex street of Type B preserves its form, independent of the time, if and only if*

$$\gamma^{-1} = \frac{\tanh \pi r + \tanh \pi(r + s) - \coth \pi(2r + s)}{\tanh \pi r - \tanh \pi(r + s) + \coth \pi s}.$$

(b) *If* ζ^0, $\| \zeta^0 \| < \infty$, *is any initial vector, the solution of the infinite linear system* (4.2), *in the case of the Type B street, is given as in Theorem* 1 *where* F *is the matrix given in Table 5.*

TABLE 5

$(8\pi b^2/\Gamma_1)f^{ij}$

Type B Double Vortex Street

i \ j	1	2	3	4	5	6	7	8
1	0	$g_6(r, s, \gamma)$	$\gamma^{-1}h_1(r)$	$-\gamma^{-1}g_1(r)$	$-\gamma^{-1}h_1(r+s)$	$\gamma^{-1}g_1(r+s)$	$h_2(2r+s)$	$-g_2(2r+s)$
2	$g_6(r, s, \gamma)$	0	$-\gamma^{-1}g_1(r)$	$-\gamma^{-1}h_1(r)$	$\gamma^{-1}g_1(r+s)$	$\gamma^{-1}h_1(r+s)$	$-g_2(2r+s)$	$-h_2(2r+s)$
3	$-\bar{h}_1(r)$	$\bar{g}_1(r)$	0	$g_7(r, s, \gamma)$	$-\gamma^{-1}h_2(s)$	$\gamma^{-1}g_2(s)$	$-\bar{h}_1(r+s)$	$-\bar{g}_1(r+s)$
4	$\bar{g}_1(r)$	$\bar{h}_1(r)$	$g_7(r, s, \gamma)$	0	$\gamma^{-1}g_2(s)$	$\gamma^{-1}h_2(s)$	$-\bar{g}_1(r+s)$	$\bar{h}_1(r+s)$
5	$-\bar{h}_1(r+s)$	$\bar{g}_1(r+s)$	$-\gamma^{-1}h_2(s)$	$-\gamma^{-1}g_2(s)$	0	$-g_7(r, s, \gamma)$	$-\bar{h}_1(r)$	$-\bar{g}_1(r)$
6	$\bar{g}_1(r+s)$	$\bar{h}_1(r+s)$	$-\gamma^{-1}g_2(s)$	$\gamma^{-1}h_2(s)$	$-g_7(r, s, \gamma)$	0	$-\bar{g}_1(r)$	$\bar{h}_1(r)$
7	$h_2(2r+s)$	$g_2(2r+s)$	$-\gamma^{-1}h_1(r+s)$	$-\gamma^{-1}g_1(r+s)$	$\gamma^{-1}h_1(r)$	$\gamma^{-1}g_1(r)$	0	$-g_6(r, s, \gamma)$
8	$g_2(2r+s)$	$-h_2(2r+s)$	$-\gamma^{-1}g_1(r+s)$	$\gamma^{-1}h_1(r+s)$	$\gamma^{-1}g_1(r)$	$-\gamma^{-1}h_1(r)$	$-g_6(r, s, \gamma)$	0

TABLE 6

Type B Double Vortex Street

$(8\pi b^2/\Gamma_1)F_1$

$h_2(2r+s)$	$g_6(r, s, \gamma) + g_2(2r+s)$	$\gamma^{-1}(h_1(r) - h_1(r+s))$	$-\gamma^{-1}(g_1(r) + g_1(r+s))$
$g_6(r, s, \gamma) - g_2(2r+s)$	$h_2(2r+s)$	$\gamma^{-1}(g_1(r+s) - g_1(r))$	$-\gamma^{-1}(h_1(r) + h_1(r+s))$
$-\bar{h}_1(r) - \bar{h}_1(r+s)$	$\bar{g}_1(r) + \bar{g}_1(r+s)$	$-\gamma^{-1}h_2(s)$	$g_7(r, s, \gamma) - \gamma^{-1}g_2(s)$
$\bar{g}_1(r) - \bar{g}_1(r+s)$	$\bar{h}_1(r) - \bar{h}_1(r+s)$	$g_7(r, s, \gamma) + \gamma^{-1}g_2(s)$	$-\gamma^{-1}h_2(s)$,

$(8\pi b^2/\Gamma_1)F_2$

$-h_2(2r+s)$	$g_6(r, s, \gamma) - g_2(2r+s)$	$\gamma^{-1}(h_1(r) + h_1(r+s))$	$-\gamma^{-1}(g_1(r) - g_1(r+s))$
$g_6(r, s, \gamma) + g_2(2r+s)$	$-h_2(2r+s)$	$-\gamma^{-1}(g_1(r) + g_1(r+s))$	$\gamma^{-1}(h_1(r+s) - h_1(r))$
$-\bar{h}_1(r) + \bar{h}_1(r+s)$	$\bar{g}_1(r) - \bar{g}_1(r+s)$	$\gamma^{-1}h_2(s)$	$g_7(r, s, \gamma) + \gamma^{-1}g_2(s)$
$\bar{g}_1(r) + \bar{g}_1(r+s)$	$\bar{h}_1(r) + \bar{h}_1(r+s)$	$g_7(r, s, \gamma) - \gamma^{-1}g_2(s)$	$\gamma^{-1}h_2(s)$.

(c) *Necessary for the stability of the system* (4.2), *in the Type B case, is that* $\alpha^2 - 4\beta \geqq 0$, $\alpha \geqq 0$, $\beta \geqq 0$, *where* α, β *are functions defined by* (4.16), (4.17).

Remarks. As $s \to +\infty$, $\gamma^{-1} \to 1$, and the relations in Theorem 6 (c) reduce to the von Kármán relation $\cosh^2 \pi r = 2$. A preliminary exploratory calculation shows that when $\cosh^2 \pi r = 2$, then $\beta < 0$ for values of πs from 0 to approximately $\pi s = 1.36$. Therefore, instability actually exists for $r = .28$, $0 < s < .43$. Landweber [4] observed a double vortex street of Type B in the wake of two adjacent parallel cylinders at values of r, s given by $r = .33$, $s = .53$.

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ON SUPERSONIC FLOW PAST A SLIGHTLY YAWING CONE. II*

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Introduction. This paper continues the analysis[1] of the flow produced by a circular cone, of not too blunt an angle, moving through air at high speed in a direction making a small angle ε (the "angle of yaw") with its axis. In [5] this flow was worked out on the assumption that ε^2 is negligible. Here our principal object is to investigate the terms in ε^2, which are also of ballistic interest; for example, we shall determine the effect of the yaw on the head drag, and the second-order effects of the yaw on the shock-wave configuration. As a by-product, it will be shown that the main differential equation of [5] is solvable in terms of quadratures.

The method is essentially the same as in [5], familiarity with which will be taken for granted. However, a complication arises (apart from the greater complexity of the equations), in that the flow exhibits some unexpected singularities, essentially because its fundamental equations are nonlinear. Thus, whereas in [5] it was possible to justify the approximations which were made there, such a justification is lacking here and the final results rest on physical assumptions which, while plausible in most cases of ballistic interest, may well break down at high Mach numbers. For numerical results we refer to the extensive tabulations in [3].

1. Expansions. As in [5], we consider the motion relative to the solid cone, and use spherical co-ordinates (r, θ, ϕ), with the axis $\theta = 0$ parallel to the flow at infinity, the axis of the cone having equation $\{\theta = \varepsilon, \phi = 0\}$. The velocity-components at the point (r, θ, ϕ), in the directions of increasing r, θ, ϕ, respectively, are u, v, w, and the pressure and density are p and ρ.[2] We are concerned with a "conical" flow, in which u, v, w, p, ρ are independent of r; and the Mach number of the undisturbed flow is assumed fixed throughout. It will be assumed throughout that ε^3 is negligible.

It was shown in [5] that $u = \bar{u} + \varepsilon x \cos\phi + o(\varepsilon)$, etc., where $\bar{u}$ and x are

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[1] See [5] in the bibliography at the end of the paper.

[2] Note that p is the *absolute* pressure, and not (as usual in ballistics) the excess of pressure over atmospheric.

[3] The basic differential equations are homogeneous in the other harmonics, essentially because they involve the first order quantities x, y, etc., only to the second degree, and these involve (apart from z) only $\cos\phi$. The boundary conditions are homogeneous in the other harmonics, essentially because the equation of the surface of the solid cone involves only $\sin^2\phi$ (cf. (4)).

certain functions of θ only; $\bar{u}$ is of course the "non-yaw" value of u when $\varepsilon = 0$. We could now expand $u - \bar{u} - \varepsilon x \cos\phi$, etc., in Fourier series in ϕ, much as was done in [5], taking for example

$$u = \bar{u} + \varepsilon x \cos\phi + \Sigma(a_n \cos n\phi + A_n \sin n\phi),$$

with similar expansions for v, w, p, ρ. However, it will follow (again, much as in [5]) that the boundary conditions can be satisfied by (and, in general, require) taking $A_m = 0$ for all m, and $a_n = 0$ unless $n = 0$ or 2. This is because the differential equations and boundary conditions to be satisfied by A_m, a_n, etc., will be of the same nature as those derived below for the non-zero terms (cf. (12)–(16) and (17)–(22)) but *homogeneous*.[3] Further, a_0 and a_2 will turn out to be of order ε^2. Hence we expand the values of u, v, etc. at the point (r, θ, ϕ), as:

$$\left.\begin{aligned} u &= \bar{u} + \varepsilon x \cos\phi + \varepsilon^2(u_0 + u_2 \cos 2\phi), \\ v &= \bar{v} + \varepsilon y \cos\phi + \varepsilon^2(v_0 + v_2 \cos 2\phi), \\ w &= \varepsilon z \sin\phi + \varepsilon^2 w_2 \sin 2\phi, \\ p &= \bar{p} + \varepsilon\eta \cos\phi + \varepsilon^2(p_0 + p_2 \cos 2\phi), \\ \rho &= \bar{\rho} + \varepsilon\xi \cos\phi + \varepsilon^2(\rho_0 + \rho_2 \cos 2\phi), \end{aligned}\right\} \tag{1}$$

where $\bar{u}$, $\bar{v}$, $\cdots$, $\bar{p}$, x, y, $\cdots$, ξ, u_0, u_2, $\cdots$, ρ_2 are functions of θ only, and where $\bar{u}$, $\cdots$, $\bar{p}$, x, $\cdots$, ξ may be regarded as known.[4] Similarly the shock-wave surface will have equation

$$\theta = \theta_w + \varepsilon\alpha \cos\phi + \varepsilon^2(\beta_0 + \beta_2 \cos 2\phi), \tag{2}$$

where θ_w, α, β_0, β_2 are constants, of which θ_w and α are known.

It will be convenient to use also a second system of spherical co-ordinates (r, Θ, Φ) in which the axis $\Theta = 0$ is the axis of the solid cone, and the planes $\phi = 0$, $\Phi = 0$, are the same (the "plane of yaw"). We shall obtain similar expansions for p, ρ, and the velocity components u, V, W in the directions of increasing r, Θ, Φ, at the point (r, Θ, Φ), say[5]

$$\left.\begin{aligned} u &= \bar{u} + \varepsilon X \cos\Phi + \varepsilon^2(U_0 + U_2 \cos 2\Phi), \\ V &= \bar{v} + \varepsilon Y \cos\Phi + \varepsilon^2(V_0 + V_2 \cos 2\Phi), \\ W &= \varepsilon Z \sin\Phi + \varepsilon^2 W_2 \sin 2\Phi, \\ p &= \bar{p} + \varepsilon \mathrm{H} \cos\Phi + \varepsilon^2(P_0 + P_2 \cos 2\Phi), \\ \rho &= \bar{\rho} + \varepsilon\,\Xi \cos\Phi + \varepsilon^2(\mathrm{P}_0 + \mathrm{P}_2 \cos 2\Phi), \end{aligned}\right\} \tag{3}$$

where here the functions $\bar{u}$, $\cdots$, $\bar{p}$, X, Y, $\cdots$, Ξ, U_0, $\cdots$, P_2 have Θ as independent variable.

It is easy to derive the relations between these co-ordinates and expansions.

[4] They are extensively tabulated in [1] and [2].

[5] The usage of capital letters here departs from the notation of [5].

For this purpose we introduce rectangular Cartesian co-ordinates $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, with the origin at the cone vertex, the $x^{(1)}$-axis along $\theta = 0$, and the $x^{(2)}$-axis in the plane of yaw (chosen so that the axis of the solid cone is in the positive quadrant), and find

$$x^{(1)}/r = \cos\theta = \cos\Theta\cos\varepsilon - \sin\Theta\cos\Phi\sin\varepsilon,$$
$$x^{(2)}/r = \sin\theta\cos\phi = \cos\Theta\sin\varepsilon + \sin\Theta\cos\Phi\cos\varepsilon,$$
$$x^{(3)}/r = \sin\theta\sin\phi = \sin\Theta\sin\Phi.$$

Thus, if $\theta = \Theta + h\varepsilon + k\varepsilon^2$ (neglecting ε^3, of course), the first of the above equations gives

$$\begin{aligned}\cos\theta &= \cos\Theta - (h\varepsilon + k\varepsilon^2)\sin\Theta - \tfrac{1}{2}h^2\varepsilon^2\cos\Theta \text{ (by Taylor's theorem)}\\ &= \cos\Theta\cdot(1 - \tfrac{1}{2}\varepsilon^2) - \varepsilon\sin\Theta\cos\Phi,\end{aligned}$$

whence

$$h = \cos\Phi, \qquad h^2 + 2k\tan\Theta = 1, \text{ so } k = \tfrac{1}{2}\sin^2\Phi\cot\Theta.$$

Similarly the third of the above equations gives ϕ in terms of Θ and Φ, and we have, for each point $(r, \theta, \phi) = (r, \Theta, \Phi)$,

$$\begin{aligned}\theta &= \Theta + \varepsilon\cos\Phi + \tfrac{1}{2}\varepsilon^2\cot\Theta\sin^2\Phi\\ \phi &= \Phi - \varepsilon\cot\Theta\sin\Phi + \tfrac{1}{2}\varepsilon^2\sin\Phi\cos\Phi\,(2\cot^2\Theta + 1).\end{aligned} \tag{4}$$

Again, if the velocity-components in the Cartesian frame are $u^{(1)}$, $u^{(2)}$, $u^{(3)}$, we obtain

$$u^{(1)} = u\cos\theta - v\sin\theta,$$
$$u^{(2)} = u\sin\theta\cos\phi + v\cos\theta\cos\phi - w\sin\phi,$$
$$u^{(3)} = u\sin\theta\sin\phi + v\cos\theta\sin\phi + w\cos\phi,$$

and

$$\begin{aligned}u = {}& u^{(1)}(\cos\Theta\cos\varepsilon - \sin\Theta\cos\Phi\sin\varepsilon)\\ &+ u^{(2)}(\cos\Theta\sin\varepsilon + \sin\Theta\cos\Phi\cos\varepsilon) + u^{(3)}\sin\Theta\sin\Phi,\\ V = {}& -u^{(1)}(\sin\Theta\cos\varepsilon + \cos\Theta\cos\Phi\sin\varepsilon)\\ &+ u^{(2)}(-\sin\Theta\sin\varepsilon + \cos\Theta\cos\Phi\cos\varepsilon) + u^{(3)}\cos\Theta\sin\Phi,\\ W = {}& u^{(1)}\sin\Phi\sin\varepsilon - u^{(2)}\sin\Phi\cos\varepsilon + u^{(3)}\cos\Phi,\end{aligned}$$

whence (from (4))

$$V = v + \varepsilon w\sin\Phi\csc\Theta - \tfrac{1}{2}\varepsilon^2 v\csc^2\Theta\sin^2\Phi \text{ and}$$
$$W = w - \varepsilon v\sin\Phi\csc\Theta + O(\varepsilon^2).$$

On equating (1) and (3), and using Taylor's theorem, we finally obtain the relations between X, x, etc., for the *same* values of the independent variables

(i.e., $\theta = \Theta$ here):

$$X = x + \bar{u}', \qquad Y = y + \bar{v}', \qquad Z = z - \bar{v}\csc\Theta,$$
$$\Xi = \xi + \bar{p}', \qquad \mathrm{H} = \eta + \bar{p}', \tag{5}$$

and, on writing $\pm$ for $(-1)^{n/2}$, where n takes the values 0 and 2 only,

$$V_n = v_n \pm \tfrac{1}{2}z\csc\Theta \mp \tfrac{1}{4}\bar{v}\csc^2\Theta \pm \tfrac{1}{4}\bar{v}'\cot\Theta + \tfrac{1}{4}\bar{v}'' + \tfrac{1}{2}y' \pm \tfrac{1}{2}y\cot\Theta$$
$$P_n = p_n \pm \tfrac{1}{2}\eta\cot\Theta + \tfrac{1}{2}\eta' \pm \tfrac{1}{4}\bar{p}'\cot\Theta + \tfrac{1}{4}\bar{p}'', \tag{6}$$

together with expressions for U_n, etc., which will not be needed. Here, as in [5], $'$ denotes differentiation with respect to the independent variable.

The ranges of θ and Θ for which the expansions (1) and (3) are assumed to be valid, will require further examination later.

2. Integration of the Equation for the First Approximation.

In [5] it was shown that x is given by a differential equaion ([5, (41)]) which can be written (with a change of notation)[6]

$$x'' + Ax' + Bx + Cd = 0, \tag{7}$$

together with certain boundary conditions ([5, (43), (45)]). Here A, B, C are known functions of θ (see [5, (38), (41)]), and d is a constant equal to

$$\eta/\bar{p} - \gamma\xi/\bar{p}.$$

Since the same reasoning applies equally well to the (r, Θ, Φ) coordinates, X must satisfy the *same* differential equation (though with different boundary conditions); note that the constant d is unchanged, since

$$\eta/\bar{p} - \gamma\xi/\bar{p} = \mathrm{H}/\bar{p} - \gamma\Xi/\bar{p}$$

from (5) and the relation $\bar{p}/\bar{\rho}^{\gamma} = $ constant.[7] Thus $X - x$ satisfies the homogenized form of (7); that is, from (5),

$$\bar{u}''' + A\bar{u}'' + B\bar{u}' = 0 \tag{8}$$

(as may also be verified directly on using [5, (1)–(5)]). It is now a routine matter to "solve" (7). We write $(x/\bar{u}')' = \zeta$, and obtain $\zeta' + \zeta(A + 2\bar{u}''/\bar{u}') = -C\,d/\bar{u}'$, i.e., $(E\zeta)' = -C\,dE/\bar{u}'$, where $E = \exp\left\{\int (A + 2\bar{u}''/\bar{u}')\,d\theta\right\}$. Thus

$$x = \bar{u}'\left[k_1 - \int_{\theta_\infty}^{\theta}\frac{d\theta}{E}\left\{\int_{\theta_\infty}^{\theta}\frac{CE\,d}{\bar{u}'}\,d\theta + k_2\right\}\right] \tag{9}$$

[6] The notation here agrees with [2] and [3]. In the notation of [5], x is x_1/ε, and the equation is $x'' + Bx' + (C - D)x = dDT/(\gamma - 1)$.

[7] The function C really depends on the boundary conditions; but $C = D(x + z\sin\theta)$ in the notation of [5], and so is also invariant, from (5). The method does not apply to the differential equation for the second approximation (30), since the constant term there involves the first order quantities and is not invariant under the change of co-ordinates.

where the constants k_1, k_2 are found from the boundary conditions [5, (43)]; simplifying by using [5, (2) and (8)], we obtain

$$k_1 = -d\bar{p}/(\gamma - 1)\bar{p}\bar{v}(\bar{u} + \bar{v}\cot\theta)$$

$$k_2 = \frac{dE\bar{p}}{\bar{v}(\gamma - 1)\bar{p}(\bar{u} + \bar{v}\cot\theta)}\left\{\frac{6\bar{u}\cot\theta}{(\gamma + 1)(\bar{u} + \bar{v}\cot\theta)} - \frac{2\bar{u} + \bar{v}\cot\theta}{\bar{v}}\right\} \quad (10)$$

all evaluated with $\theta = \theta_w$. Finally, on using the relations between the non-yaw quantities [5, (1)–(5)], we find (to within an immaterial multiplicative constant)

$$E = \bar{v}^3 \csc^5\theta/\bar{p}\bar{p}'(\bar{u} + \bar{v}\cot\theta)^2 \quad (11)$$

In a sense, equations (9)–(11) completely solve the problem, to the first approximation considered in [5]; but they are not well adapted to numerical computation, since the quantity between square brackets in (9) becomes infinite when $\theta = \theta_s$ (the semi-angle of the solid cone), because $\bar{u}'$ $(= \bar{v})$ then vanishes.

3. The differential equations. Returning to the second approximation, the fundamental equations are: the first and third of the equations of motion [5, (11)] (the second equation of motion is conveniently replaced by the Bernoulli equation), viz.

$$v\,\partial u/\partial\theta + w\csc\theta\,\partial u/\partial\phi - v^2 - w^2 = 0,$$

$$v\,\partial w/\partial\theta + w\csc\theta\,\partial w/\partial\theta + (\partial p/\partial\phi)/\rho\sin\theta + w(u + v\cot\theta) = 0,$$

the equation of continuity ([5, (12)]),

$$2\rho u\sin\theta + \partial(\rho v\sin\theta)/\partial\theta + \partial(\rho w)/\partial\phi = 0,$$

the Bernoulli equation ([5, (14)])

$$\tfrac{1}{2}(u^2 + v^2 + w^2) + \gamma p/(\gamma - 1)\rho = \text{constant},$$

and the adiabatic equation (cf. argument leading to [5, (13)],

$$\gamma p/\rho = \left\{v\frac{\partial p}{\partial\theta} + w\csc\theta\frac{\partial p}{\partial\phi}\right\}\Big/\left\{v\frac{\partial\rho}{\partial\theta} + w\csc\theta\frac{\partial\rho}{\partial\phi}\right\}.$$

We substitute from (1) in these equations; the terms independent of ε, or of order ε, will of course cancel; and, on equating the coefficients of $\cos n\phi$ in the terms of order ε^2, we obtain the equations which follow.[8]

$$u_n' - v_n = \pm zt\csc\theta/2\bar{u}' \quad (12)$$

where $t = x + z\sin\theta$ (cf. [5, (38)]),

$$\bar{v}w_2' + (\bar{u} + \bar{v}\cot\theta)w_2 - 2p_2/\bar{p}\sin\theta$$
$$+ \tfrac{1}{2}\{yz' + z(x + y\cot\theta + z\csc\theta) + \xi\eta/\bar{p}^2\sin\theta\} = 0 \quad (13)$$

[8] Throughout, n takes the values 0 and 2 only, and $\pm$ means $(-1)^{n/2}$.

$$v_n' + v_n(\cot\theta + \bar{p}'/\bar{p}) + 2u_n + nw_n \csc\theta + v(\rho_n/\bar{p})' + \tfrac{1}{2}(y - \bar{v}\xi/\bar{p})(\xi/\bar{p})' \mp \xi z/2\bar{p}\sin\theta = 0 \quad (14)$$

$$\bar{u}u_n + \bar{v}v_n + \tfrac{1}{4}(x^2 + y^2 \pm z^2) + \frac{\gamma}{\gamma - 1}\frac{\bar{p}}{\bar{\rho}}\left\{\frac{p_n}{\bar{p}} - \frac{\rho_n}{\bar{\rho}} + \tfrac{1}{2}\frac{\xi}{\bar{p}}\left(\frac{\xi}{\bar{p}} - \frac{\eta}{\bar{p}}\right)\right\} = 0 \quad (15)$$

and

$$\bar{v}(\gamma\bar{\rho}'p_n - p_n'\bar{\rho} + \gamma\rho_n'\bar{p} - \bar{p}'\rho_n) = \tfrac{1}{2}\{y(\eta'\bar{p} + p'\xi - \gamma\xi'\bar{p} - \gamma\bar{p}'\eta) \pm z\csc\theta(\gamma\xi\bar{p} - \eta\bar{p}) + \bar{v}(\eta'\xi - \gamma\xi'\eta)\}.$$

In this last equation, the relations [5; (3), (19)] show first that the expression multiplied by y vanishes, and then enable the equation to be integrated, giving

$$-p_n/\bar{p} + \gamma\rho_n/\rho + \tfrac{1}{4}(\eta/\bar{p})^2 - \gamma(\xi/\bar{p})^2/4 \pm \tfrac{1}{2}d\int_{\theta_w}^{\theta}\frac{z\,d\theta}{\bar{v}\sin\theta} + e_n = 0 \quad (16)$$

where e_0 and e_2 are constants.

Essentially the same equations will, of course, hold for U_n, V_n, etc.

In deriving these equations, we have however neglected not merely ε^3 but $\varepsilon^3/\bar{u}'$; and since $\bar{u}'$ vanishes at θ_s, the procedure is invalid near the surface of the solid cone—as is shown by the fact that terms in (12) and (16) become infinite at θ_s. We shall therefore *define* u_n, v_n, etc., by the equations (12)–(16), together with the boundary conditions to be considered below, and similarly define U_n, V_n, etc., by the analogous equations, so that (6) continues to hold. The expansions (1) and (3) will then be assumed to be (approximately) valid *except* in the neighbourhood of the solid cone. More precisely, we assume that (3) holds in the region between the shock-wave and the conical surface $\Theta = \theta^*$, where $\theta^* > \theta_s$ and $\theta^* - \theta_s$ is very small. This assumption is plausible, in most cases of ballistic interest, since the divergent terms in (12) and (16) are multiplied by the quantities t and d, which will be very small for all but strong shocks (cf. [5, §8] and the tabulated values in [2]). We shall return to this point later, showing that the applications made in §8 below are not sensitive to the particular value adopted for θ^*.[9]

4. Boundary conditions. (a) *At the shock-wave.* These are derived from the Rankine-Hugoniot equations just as in [5, §4]. The only point requiring care is that the normal vector n to the shockwave surface (cf. [5, (26)]) is no longer a unit-vector to the present approximation; in fact,

$$\underline{n} = \nabla\{\theta - \theta_w - \varepsilon\alpha\cos\phi - \varepsilon^2(\beta_0 + \beta_2\cos 2\phi)\}$$
$$= \{0,\ 1,\ \csc\theta(\varepsilon\alpha\sin\phi + 2\varepsilon^2\beta_2\sin 2\phi)\},$$

so that $|\underline{n}| = 1 + \tfrac{1}{2}\varepsilon^2\alpha^2\csc^2\theta\sin^2\phi$. The unit tangent vector $\underline{t}_2$ of [5, (28)] needs modification accordingly. As before, we substitute from (1) in [5, (21)–(25)],

[9] In confirmation of the assumption made here, the divergence of the infinite terms is so slow that in [3] reasonable values for ρ_n and u_n at θ_s are obtained by extrapolation, though these quantities are theoretically infinite.

using (2) and Taylor's theorem to express all quantities in terms of values when $\theta = \theta_w$; and the terms in ε^2 give:
When $\theta = \theta_w$,

$$\beta_n(\bar{u}' + \bar{u}\tan\theta) + u_n + \tfrac{1}{4}\alpha^2(\bar{u} + \bar{u}'') + \tfrac{1}{2}\alpha x' = 0,$$

$$2\beta_2(\bar{u}' + \bar{u}\tan\theta) - w_2\sin\theta + \tfrac{1}{4}\alpha^2(\bar{u}'' + \bar{u}) + \tfrac{1}{2}\alpha(y - z\cos\theta - z'\sin\theta) = 0,$$

$$\beta_n(\bar{v}' - \bar{v}\cot\theta + \bar{v}\bar{p}'/\bar{p}) + \bar{v}\bar{p}_n/\bar{p} + v_n + \tfrac{1}{2}y\xi/\bar{p} + \tfrac{1}{2}\alpha(y' \pm z\csc\theta + (y\bar{p}' + \bar{v}'\xi + \bar{v}\xi')/\bar{p}) + \tfrac{1}{4}\alpha^2(\bar{v}'' + \bar{v} + (2\bar{v}'\bar{p}' + \bar{v}\bar{p}'')/\bar{p}) = 0,$$

$$\beta_n\{\bar{p}'/\bar{p}\bar{u}' + \bar{u}'' + \bar{u}'\cot\theta + 2\bar{u}\} + p_n/\bar{p}\bar{u}' + v_n - \tfrac{1}{4}\alpha^2\{(\bar{u}\tan\theta + \bar{u}')\cdot(1 \pm 2\csc^2\theta) - 2(\bar{u}'' + \bar{u})\cot\theta - \frac{\bar{p}''}{\bar{p}\bar{u}'} - \bar{v}'' + \bar{u}\tan\theta\} + \tfrac{1}{2}\alpha\{\eta'/\bar{p}\bar{u}' + y' \pm z\csc\theta + y\cot\theta\} = 0,$$

$$\beta_n\{\bar{v}' + \bar{v}\cot\theta + 2(\gamma - 1)\bar{u}/(\gamma + 1)\} + v_n + \tfrac{1}{4}\alpha^2\{\bar{v}'' + \bar{v} - 2(1 \pm 1)\cdot[\bar{v} + ((\gamma - 1)/(\gamma + 1))\bar{u}\tan\theta]\csc^2\theta\} + \tfrac{1}{2}\alpha(y' \pm z\csc\theta) = 0.$$

These equations can now be simplified (a process which uses most of the equations in [5]), and the first of them can be used to eliminate β_n from the others. This gives:
When $\theta = \theta_w$,

$$(\bar{u}' + u\tan\theta)\beta_n + u_n = \alpha^2\{(\gamma - 7)\bar{u} + (\gamma + 3)\bar{u}'\cot\theta\}/4(\gamma + 1) \tag{17}$$

$$2u_2 + w_2\sin\theta = \alpha^2\tan\theta(\bar{u} + \bar{u}'\cot\theta)^2/2\bar{u}' \tag{18}$$

$$2u_0\cot\theta + v_0 + \bar{u}'\rho_0/\bar{p} = -\alpha^2(\bar{u}\tan\theta + \bar{u}') + [\alpha/(\gamma + 1)]^2\cdot\{2(\gamma - 1)^2\bar{u}^2/\bar{u}' + (3\gamma^2 + \gamma - 10)\bar{u}\cot\theta + (\gamma^2 + 5\gamma + 6)\bar{u}'\cot^2\theta\} \tag{19$_0$}$$

$$2u_2\cot\theta + v_2 + \bar{u}'\rho_2/\bar{p} = [\alpha/(\gamma + 1)]^2\{2(\gamma - 1)^2\bar{u}^2/\bar{u}' + (4\gamma^2 + 3\gamma - 9)\bar{u}\cot\theta + (2\gamma^2 + 7\gamma + 7)\bar{u}'\cot^2\theta\} \tag{19$_2$}$$

$$-u_n\cot\theta + v_n + p_n/\bar{\rho}\bar{u}' = -\alpha^2\{(\gamma + 1)\bar{u}^2/\bar{u}' + 2\gamma\bar{u}\cot\theta + (\gamma - 1)\bar{u}'\cot^2\theta\}/2(\gamma + 1) \tag{20}$$

$$(4\bar{u} - 2\bar{u}'\cot\theta)u_0/(\gamma + 1) + (\bar{u}\tan\theta + \bar{u}')v_0 = -2\alpha^2\bar{u}\tan\theta(\bar{u}\tan\theta + \bar{u}')/(\gamma + 1) + \alpha^2\{(\gamma^2 + 4\gamma - 21)\bar{u}^2 + (2\gamma^2 + 4\gamma + 14)\bar{u}\bar{u}'\cot\theta + (\gamma^2 - 1)\bar{u}'^2\cot^2\theta\}/2(\gamma + 1)^2 \tag{21$_0$}$$

$$(4\bar{u} - 2\bar{u}'\cot\theta)u_2/(\gamma + 1) + (\bar{u}\tan\theta + \bar{u}')v_2 = \alpha^2\{(\gamma^2 + 8\gamma - 17)\bar{u}^2 + (2\gamma^2 + 8\gamma + 18)\bar{u}\bar{u}'\cot\theta + (\gamma^2 - 1)\bar{u}'^2\cot^2\theta\}/2(\gamma + 1)^2 \tag{21$_2$}$$

(b) *At the solid cone* Here the normal velocity, V, is zero. Now if temporarily we return to the original definition of V_n, (3) shows that $V_n = 0$ when $\Theta = \theta_s$, so that when $\theta = \theta^*$, where the two definitions approximately coincide, V_n is

presumably small. Thus the right-hand side of the first of equations (6) is small when $\Theta = \theta^*$, and it is natural to assume that it remains small at θ_s.[10] We therefore take the boundary condition to be that, when $\theta = \theta_s$,

$$v_n \pm \tfrac{1}{2}z \csc\theta \mp \tfrac{1}{4}\bar{v}\csc^2\theta \pm \tfrac{1}{4}\bar{v}'\cot\theta + \tfrac{1}{4}\bar{v}'' + \tfrac{1}{2}y' \pm \tfrac{1}{2}y\cot\theta = 0.$$

On simplification by means of equations 1, 2, 5, 6, 18, 35 of [5], this reduces to:

When $\theta = \theta_s$, $\quad v_0 = x \quad$ and

$$v_2 = \bar{u}\cot\theta + x + z\csc\theta \tag{22}$$

5. Reduction of the equations. To avoid singularities at θ_s, we write[11]

$$s_n = (u_n)_w + \int_{\theta_w}^{\theta} v_n\, d\theta = u_n \mp R,$$

where (from (12))

$$R = \int_{\theta_w}^{\theta} zt\, d\theta/2\bar{v}\sin\theta \tag{23}$$

Thus

$$s_n' = v_n, \qquad (s_n)_w = (u_n)_w. \tag{24}$$

We shall convert (14) into a differential equation for s_n. The first step is to evaluate w_2. We write (cf. [5, (36)])

$$\tau_2 = 2s_2 + w_2\sin\theta \tag{25}$$

and eliminate p_2 and ρ_2 between (13), (15) and (16) (by adding (15), $\frac{1}{2}\sin\theta$ times (13), and $\bar{p}/(\gamma-1)\bar{\rho}$ times (16)). The result can be written, after simplification by means of [5, (17), (19), (20)],

$$\bar{u}'\tau_2' + \bar{u}\tau_2 = S - 2e_2\bar{p}/(\gamma-1)\bar{\rho},$$

where

$$S = 2\bar{u}R + \frac{1}{2}t\left(\frac{y\bar{u}}{\bar{v}} - x\right) + \frac{dp}{2(\gamma-1)\bar{\rho}}\left\{4Q + \frac{y}{\bar{v}} - \frac{(\gamma-1)\xi}{\bar{p}} - d\right\}, \tag{26}$$

and

$$Q = \int_{\theta_w}^{\theta} (z/2\bar{v}\sin\theta)\, d\theta$$

The integration of (26) follows on multiplying through by

$$(-\bar{v}\bar{\rho}\sin\theta)^{-\frac{1}{2}}/\bar{v} \text{ (cf. [5, §6])}, \text{ giving}$$

$$\frac{\tau_2}{\sqrt{(-\bar{v}\bar{\rho}\sin\theta)}} = \text{constant} + \int_{\theta_w}^{\theta} \frac{S\, d\theta}{\bar{v}\sqrt{(-\bar{v}\bar{\rho}\sin\theta)}} - \frac{2e_2}{\gamma-1}\int_{\theta_w}^{\theta} \frac{\bar{p}\, d\theta}{\bar{v}\rho\sqrt{(-\bar{v}\bar{\rho}\sin\theta)}}$$

[10] This assumption will be justified at the end of §7 below.

[11] The notation $(u_n)_w$ means the value of u_n when $\theta = \theta_w$; similarly, e.g., $(s_n)_s$ means the value of s_n when $\theta = \theta_s$.

The last integral on the right is a numerical multiple of $t(= t_1/\varepsilon$ in the notation of [5]; see [5, (38)]), and the constant of integration is determined by (18). Thus

$$\tau_2 = \sigma + 2e_2 t/d, \quad \text{where} \quad t = -\frac{d}{\gamma - 1}(-\bar{v}\bar{p}\sin\theta)^{\frac{1}{2}} \int_{\theta_w}^{\theta} \frac{\bar{p}\, d\theta}{\bar{v}\bar{p}\sqrt{(-\bar{v}\bar{p}\sin\theta)}}$$

and

$$\sigma = \alpha^2 \tan\theta_w (\bar{u}_w + \bar{v}_w \cot\theta_w)^2 \cdot \left\{\frac{\bar{v}\bar{p}\sin\theta}{\bar{v}_w \bar{p}_w \sin\theta_w}\right\}^{\frac{1}{2}} \Big/ 2\bar{v}_w + (-\bar{v}\bar{p}\sin\theta)^{\frac{1}{2}} \int_{\theta_w}^{\theta} \frac{S\, d\theta}{\bar{v}\sqrt{(-\bar{v}\bar{p}\sin\theta)}} \tag{27}$$

and, from (25), we now have

$$w_2 = (\sigma + 2e_2 t/d - 2s_2)\csc\theta \tag{28}$$

Next we evaluate $(\rho_n/\bar{\rho})'$. From (15), (16) and [5, (19)],

$$-\rho_n/\bar{\rho} = \frac{\bar{\rho}}{\gamma\bar{p}}\left\{\bar{u}u_n + \bar{v}v_n + \frac{1}{4}(x^2 + y^2 \pm z^2)\right\} + \frac{\gamma - 2}{4}\left(\frac{\xi}{\bar{\rho}}\right)^2 + \frac{d}{2}\left(\frac{\xi}{\bar{\rho}}\right) \pm \frac{d}{\gamma - 1} Q + \text{constant},$$

from which it is easy to express $(\rho_n/\bar{\rho})'$ in terms of s_n and its derivatives. We find, after some simplification,

$$\begin{aligned} -(\rho_n/\bar{\rho})' = \frac{\bar{\rho}}{\gamma\bar{p}}\Big[&\bar{v}s_n'' + \{\bar{v}' - (\gamma - 1)\bar{v}\bar{p}'/\bar{p} + \bar{u}\}s_n' \\ &+ \{\bar{v} - (\gamma - 1)\bar{u}\bar{p}'/\bar{p}\}(s_n \pm R) + \frac{1}{2}(xx' + yy' \pm zz') \\ &- (\gamma - 1)(x^2 + y^2 \pm z^2)\bar{p}'/4\bar{p} \pm \frac{\bar{u}zt}{2\bar{v}\sin\theta}\Big] \\ &+ \frac{1}{2}\left\{(\gamma - 2)\frac{\xi}{\bar{\rho}} + d\right\}\left(\frac{\xi}{\bar{\rho}}\right)' \pm dz/2(\gamma - 1)\bar{v}\sin\theta \end{aligned} \tag{29}$$

Finally we substitute from (28) and (29) in (14) to obtain the desired differential equation for s_n. It is simplified much as in [5, §6], and can then be written

$$s_n'' + As_n' + \{B + (1 - n^2)Cd/t\}s_n \pm (B + Cd/t)R + nC(\sigma d/t + 2e_2) + D_n = 0, \tag{30}$$

where A, B, C are the same as in (7) above (i.e., are B, C–D and Dt/d in the notation of [5, (41)]); that is,

$$\begin{aligned} A &= \cot\theta + \lambda\{2\bar{u} + 3\bar{v}\cot\theta + (\gamma + 1)\lambda\bar{v}(\bar{u} + \bar{v}\cot\theta)\}, \\ B &= 1 - \cot^2\theta + \lambda\{-\bar{v}\cot^2\theta + (\gamma - 1)\lambda\bar{u}(\bar{u} + \bar{v}\cot\theta)\}, \\ C &= (t/d)(1 + \lambda\bar{v})\csc^2\theta, \end{aligned}$$

where

$$\lambda = \bar{p}'/\bar{p}(\bar{u} + \bar{v}\cot\theta) = 2\bar{v}/\{(\gamma-1)(c^2-\bar{u}^2) - (\gamma+1)\bar{v}^2\} = \bar{v}/(a^2-\bar{v}^2),$$

a here denoting the local sound velocity $\sqrt{\gamma\bar{p}/\bar{\rho}}$ (and not, as in [5], the velocity of sound in the undisturbed flow). Further, in the first instance we find

$$2D_n = -\lambda\{xx' + yy' \pm zz' - (\gamma-1)(x^2+y^2\pm z^2)\bar{\rho}'/2\bar{\rho} \pm \bar{u}zt/\bar{v}\sin\theta\}$$
$$-(1+\lambda\bar{v})\left[\left(\frac{\xi}{\bar{\rho}}\right)'\left\{(\gamma-1)\bar{v}\,\frac{\xi}{\bar{\rho}} + d\bar{v} - y\right\} \pm z\csc\theta\left(\frac{d}{\gamma-1} + \frac{\xi}{\bar{\rho}}\right)\right]$$

This can be converted to a form more suitable for computation, by using the results of [5] to express the first-order quantities and their derivatives in terms of x, x' $(= y)$, x'', t and z $(= (t-x)\csc\theta)$. In this way we find (with some trouble) that

$$2D_n = \tfrac{1}{2}(\gamma-1)\lambda^2(x^2+y^2)(\bar{u}+\bar{v}\cot\theta) -$$
$$(1/a^2)\cdot[\bar{v}x''\{(2+(\gamma+1)\lambda\bar{v})y + (\gamma-1)\lambda\bar{u}x\} - y^2(\bar{u}+\bar{v}\cot\theta)(1+\gamma\lambda\bar{v})^2$$
$$+ xy\{2\bar{v} + (\gamma+1)\lambda\bar{v}^2 - 2(\gamma-1)\bar{u}\lambda(1+\gamma\lambda\bar{v})(\bar{u}+\bar{v}\cot\theta)\} \qquad (31)$$
$$+ (\gamma-1)\lambda\bar{u}x^2\{\bar{v} - (\gamma-1)\bar{u}\lambda(\bar{u}+\bar{v}\cot\theta)\}] \pm$$
$$z\csc\theta\cdot\left[\frac{d(1+\lambda\bar{v})}{\gamma(\gamma-1)} + \frac{\bar{u}x}{a^2-\bar{v}^2} + \lambda\{2y + z(\cos\theta + \tfrac{1}{2}(\gamma-1)\lambda(\bar{u}\sin\theta + \bar{v}\cos\theta)\}\right]$$

Next, the values $(u_n)_w$ $(= (s_n)_w)$ and $(v_n)_w$ $(= (s_n')_w)$ of u_n and v_n when $\theta = \theta_w$ can be found from (19), (20) and (21), on using (16) to eliminate p_n and ρ_n. To illustrate the process, we sketch the case $n = 2$. By adding $\bar{p}/\bar{\rho}$ times (16), $\gamma\bar{p}/\bar{\rho}\bar{v}$ times (19_2) and $-\bar{v}$ times (20), we obtain, for $\theta = \theta_w$,

$$u_2\left(\bar{v}\cot\theta + \frac{2\gamma\bar{p}}{\bar{s}\bar{v}}\cot\theta\right) + v_2(-\bar{v} + \gamma\bar{p}/\bar{\rho}\bar{v})$$
$$= \frac{\bar{p}}{\bar{s}\bar{v}}\left[\frac{\alpha^2\gamma}{(\gamma+1)^2}\left\{2(\gamma-1)^2\frac{\bar{u}^2}{\bar{v}} + (4\gamma^2+3\gamma-9)\bar{u}\cot\theta\right.\right.$$
$$\left.\left. + (2\gamma^2+7\gamma+7)\bar{v}\cot^2\theta + \frac{\bar{v}}{4}\left\{\left(\frac{\eta}{\bar{\rho}}\right)^2 - \gamma\left(\frac{\xi}{\bar{\rho}}\right)^2 + 4e_2\right\}\right]\right.$$
$$+ \frac{\alpha^2}{2(\gamma+1)}\{(\gamma+1)\bar{u} + (\gamma-1)\bar{v}\cot\theta\}(\bar{u}+\bar{v}\cot\theta)$$

which, on using the boundary conditions satisfied by the non-yaw and first order quantities ([5, (7), (8), (30)–(33), (43)]), can be written

$$2u_2\cot\theta\{(\gamma+1)\bar{u} + (\gamma-2)\bar{v}\cot\theta\} + v_2(\gamma+1)(\bar{u}+\bar{v}\cot\theta)$$
$$= \{(\gamma+1)\bar{u} + (\gamma-1)\bar{v}\cot\theta\}\left[\frac{\alpha(\gamma-1)}{\gamma+1}\left(\bar{u}\,\frac{\xi}{\bar{\rho}} - 2x\cot\theta\right) -\right.$$
$$\left.\frac{2\alpha^2\cot\theta}{(\gamma+1)^2}\cdot\{(\gamma+1)\bar{u} - (3\gamma+4)\bar{v}\cot\theta\} + \frac{\bar{v}}{4\gamma}\left\{\left(\frac{\eta}{\bar{\rho}}\right)^2 - \gamma\left(\frac{\xi}{\bar{\rho}}\right)^2 + 4e_2\right\}\right]$$

This and (20) give simultaneous linear equations for $(u_2)_w$ and $(v_2)_w$. In this way we find (after some arduous simplification).

When $\theta = \theta_w$,

$$s_n = F_n + e_n x,\ s_n' = H_n + e_n y, \tag{32}$$

where

$$F_0 = \frac{\gamma x}{\bar{v}}\left[\frac{\bar{v}}{4}\left\{\frac{1}{\gamma}\left(\frac{\eta}{\bar{p}}\right)^2 + \left(\frac{\xi}{\bar{p}}\right)^2\right\} - \alpha^2(\bar{u} + \bar{v}\cot\theta)\left(\tan\theta + \frac{\gamma+3}{\gamma+1}\cot\theta\right)\right] \tag{33$_0$}$$
$$- \frac{\alpha^2}{4(\gamma-1)}\left\{(\gamma+3)\bar{u} + (\gamma-1)\bar{v}\cot\theta + 2\bar{u}\tan\theta\left(\frac{3y}{x} - 2\tan\theta\right)\right\}$$

$$F_2 = \frac{\gamma x}{\bar{v}}\left[\frac{\bar{v}}{4}\left\{\frac{1}{\gamma}\left(\frac{\eta}{\bar{p}}\right)^2 + \left(\frac{\xi}{\bar{p}}\right)^2\right\} - \frac{2\alpha^2}{\gamma+1}\cot\theta\,(\bar{u} + \bar{v}\cot\theta)\right] \tag{33$_2$}$$
$$- \frac{\alpha^2}{4(\gamma-1)}\left\{\frac{6y}{x}\,\bar{u}\tan\theta + (\gamma+7)\bar{u} + (\gamma-1)\bar{v}\cot\theta\right\}$$

$$H_0 = yF_0/x + \frac{\alpha^2\cot\theta}{2(\gamma+1)}\{2\bar{u}\tan\theta\left(\frac{3y}{x} - 2\tan\theta\right)$$
$$+ (\gamma+3)\bar{u} + (\gamma-1)\bar{v}\cot\theta\}$$
$$H_2 = yF_2/x + \frac{\alpha^2\cot\theta}{2(\gamma+1)}\left\{\frac{6y}{x}\,\bar{u}\tan\theta + (\gamma+7)\bar{u} + (\gamma-1)\bar{v}\cot\theta\right\} \tag{34}$$

6 The problem for computation. In the differential equation (30) and boundary conditions (32), all the coefficients, except for the constants e_0, e_2, may be regarded as known. To determine e_0 and e_2, we define auxilliary functions f_n, g_n $(n = 0, 2)$ of θ in such a way that

$$s_n = f_n + e_n g_n. \tag{35}$$

This can clearly be achieved by taking

$$\begin{cases} f_0'' + Af_0' + (B + Cd/t)(f_0 + R) + D_0 = 0 \\ \text{When } \theta = \theta_w,\ f_0 = F_0,\ f_0' = H_0 \end{cases} \tag{36$_0$}$$

$$\begin{cases} g_0'' + Ag_0' + (B + Cd/t)g_0 = 0 \\ \text{When } \theta = \theta_w,\ g_0 = x,\ g_0' = y \end{cases} \tag{37$_0$}$$

$$\begin{cases} f_2'' + Af_2' + (B - 3Cd/t)f_2 - (B + Cd/t)R + 2C\sigma d/t + D_2 = 0 \\ \text{When } \theta = \theta_w,\ f_2 = F_2,\ f_2' = H_2 \end{cases} \tag{36$_2$}$$

$$\begin{cases} g_2'' + Ag_2' + (B - 3Cd/t)g_2 + 4C = 0 \\ \text{When } \theta = \theta_w,\ g_2 = x,\ g_2' = y. \end{cases} \tag{37$_2$}$$

Now, from (22) and (24), we have:

When $\theta = \theta_s$, $\qquad e_0 = (x - f_0')/g_0'$

and

$$e_2 = (\bar{u}\cot\theta + x + z\csc\theta - f_2')/g_2' \tag{38}$$

It is now practicable (though very laborious) to compute f_n and g_n, in a numerical case, from (36) and (37), and thus to determine e_0 and e_2, and thence s_0, s_2, and the whole flow. It should be remarked that the constant terms in (36) become infinite at θ_s; in fact one can show that they are of order $(\theta - \theta_s)^{-\frac{1}{2}}$ as $\theta \to \theta_s$. Thus f_n'' will be infinite at θ_s, though f_n' and f_n will be finite and continuous there.

We can now to some extent check the assumptions on which (22) (and so (38)) was based. The argument leading to (22) shows that, under plausible physical hypotheses, (38) holds approximately when $\theta = \theta^* \fallingdotseq \theta_s$. Since the quantities appearing on the right in (38) are continuous at θ_s, (38) will continue to hold approximately when $\theta = \theta_s$ also, provided that the denominators g_n' are not too small at θ_s—a point which is confirmed by the computations for all but large Mach numbers.[12]

7. Applications. (a) *Effect on the shock-wave.* As can be seen from (2), the second-order effect of the yaw on the shock-wave surface (which remains a circular cone, yawed through an angle $\varepsilon\alpha$, if ε^2 is neglected) is to make it somewhat "elliptical."[13] The quantities β_0, β_2 are readily computable from (17), once s_0 and s_2 are found. The computed values of β_0 and β_2 agree with experience in predicting that this effect is small; it is probably too small to be capable of giving an accurate check on the theory at present.

(b) *The yaw-drag coefficient.* In order to determine the resultant pressure on the surface of the solid cone, we first show that p_n is continuous at θ_s. We shall consider only p_0 explicitly. Eliminating ρ_n between (15) and (16), we express p_0 in terms of u_0 and v_0, and thence (using (23)) in terms of s_0 and s_0'. The result (simplified by means of [5, (19)]) is

$$\begin{aligned} -p_0/\bar{p} = (\bar{p}/\bar{p})\{\bar{u}s_0 + \bar{v}s_0' + \tfrac{1}{4}(x^2 + y^2 + z^2)\} - (\eta/\bar{p})[\eta/\bar{p} - 2d]/4\gamma \\ + (d^2 + 4\gamma e_0)/4\gamma(\gamma - 1) + [\bar{u}\bar{p}R/\bar{p} + dQ/(\gamma - 1)]. \end{aligned} \tag{39}$$

The integrals Q and R occurring here are both infinite at θ_s; but, on using [5, (39)], the bracket [ ] in (39) can be written

$$\frac{\bar{p}\bar{u}}{2\bar{p}}\int_{\theta_w}^{\theta} \frac{zt\,d\theta}{\bar{v}\sin\theta} + \frac{d}{\gamma - 1}\int_{\theta_w}^{\theta} \frac{z\,d\theta}{2\bar{v}\sin\theta}$$

$$= \frac{\bar{p}\bar{u}}{2\bar{p}}\left\{\int_{\theta_w}^{\theta} \frac{z(t - t_s)\,d\theta}{\bar{v}\sin\theta} + \frac{d}{\gamma - 1}\left(\frac{\bar{p}}{\bar{p}\bar{u}} - \frac{\bar{p}_s}{\bar{p}_s\bar{u}_s}\right)\int_{\theta_w}^{\theta} \frac{z\,d\theta}{2\bar{v}\sin\theta}\right\}$$

Now, from [5, (38)], it can be shown that $t - t_s$ is of order $(\theta - \theta_s)^{\frac{1}{2}}$ as $\theta \to \theta_s$, while v is of order $(\theta - \theta_s)$. Hence the bracket [ ] converges to the finite limit

$$\frac{\bar{p}_s\bar{u}_s}{2\bar{p}_s}\int_{\theta_w}^{\theta} \frac{z(t - t_s)}{\bar{v}\sin\theta}\,d\theta;$$

[12] Indirectly. The values of g' are not given in [3], but the fractions e_0 and e_2 in (38) are small, except for very large Mach numbers.

[13] The cross-section, however, is not a true ellipse but a curve of the sixth degree.

and, on defining $(p_0)_s = \lim_{\theta \to \theta_s} p_0$, we obtain that p_0 is continuous at θ_s. Further, (39) gives (if we use [5, (6), (20)] to express $\eta_s/\bar{p}_s$ in terms of x_s):

When $\theta = \theta_s$,

$$\frac{p_0}{\bar{p}} = \frac{\gamma}{2a^2}\left\{\frac{\bar{u}^2x^2}{2a^2} + \frac{\bar{u}xd}{\gamma - 1} - 2\bar{u}s_0 - \tfrac{1}{2}(x^2 + y^2 + z^2) - \bar{u}\int_{\theta_\infty}^{\theta} \frac{z(t - t_s)}{\bar{v}\sin\theta}d\theta\right\} + \frac{d^2}{4(\gamma - 1)^2} - \frac{e_0}{\gamma - 1}. \tag{40}$$

Similarly p_2, and so P_0 and P_2, will be continuous at θ_s. Further, equation (6) simplifies considerably when $\theta = \theta_s$, since $\bar{v}$, $\bar{p}'$, $\bar{\rho}'$ vanish (cf. [5, (6)]). Thus, from [5, (2), (16)] we have—$\bar{u}y + \eta'/\bar{p} = 0$ there, so that $\eta' = \bar{\rho}\bar{u}y = 2\bar{\rho}\bar{u}^2$ at θ_s (using [5, 35]). And, from [5, (1), (2)], $\bar{p}''/\bar{p} = -\bar{u}''(\bar{u} + \bar{u}'') = -2\bar{u}^2$ at θ_s. Hence we find:

When $\theta = \theta_s$,

$$P_0/\bar{p} = (p_0/\bar{p}) + (\gamma/2a^2)(\bar{u}^2 - \bar{u}x\cot\theta) - (d/2(\gamma - 1))\cot\theta \tag{41}$$

Now consider an actual projectile having a conical head of semi-angle θ_s, slant length R and diameter D ($= 2R\sin\theta_s$), moving through air at small yaw ε. If its Mach number is sufficiently greater than 1, the flow relative to the cone will be locally supersonic and approximately steady, and so the preceding theory should apply in the neighbourhood of the conical head. The component of the force due to air pressure on the conical head, parallel to the axis of the cone, will then be

$$\int_0^R \int_0^{2\pi} pr\sin^2\theta_s\, dr\, d\Phi = \tfrac{1}{4}\pi D^2(\bar{p}_s + \varepsilon^2(P_0)_s),$$

provided that the expansion (3) for p is valid at θ_s. This should be the case, at least approximately, since the expansion (3) is assumed valid at $\theta^* \fallingdotseq \theta_s$, and P_0, P_2 are continuous, and so nearly the same at θ_s as at θ^*.

The component of the air resistance parallel to the direction of motion of the cone, is obtained by combining the above component with the component, of magnitude $\frac{1}{8}\pi D^2\varepsilon\eta_s\cot\theta_s$ (cf. [5, (54)]), perpendicular to the axis, and is found to be

$$\tfrac{1}{4}\pi D^2\bar{p}\{1 + \varepsilon^2[(P_0/\bar{p}) - \tfrac{1}{2} + (\eta/2\bar{p})\cot\theta]\},$$

evaluated with $\theta = \theta_s$. Since $\eta_s/\bar{p}_s = -(\gamma\bar{u}x/a^2 + d/(\gamma - 1))$ (cf. [5, (56)]), this result can be expressed[14] by saying that the "head drag" has its non-yaw value multiplied by $1 + \varepsilon^2 K_{DYH}$, where

$$\begin{aligned} K_{DYH} &= P_0/\bar{p} - \tfrac{1}{2}[(\gamma\bar{u}x/a^2)\cot\theta + (d/(\gamma - 1))\cot\theta + 1] \\ &= p_0/\bar{p} + (\gamma/2a^2)(\bar{u}^2 - \bar{u}x\cot\theta) - [d/(\gamma - 1)]\cot\theta - \tfrac{1}{2}. \end{aligned} \qquad (\theta = \theta_s) \tag{42}$$

[14] As was pointed out to the author by Dr. A. C. Charters, a form more convenient for applications is: increase in head drag coefficient $= K'_{DH}\varepsilon^2$, where $K'_{DH} = K_{D_0}K_{DYH} = K_{NH} - K_{D_0}/2 + \pi(P_0)_s/4\rho_1U^2$, and K_{D_0} is the head drag coefficient based on absolute pressures. Thus K_{D_0} is related to the usual head drag coefficient K_D (tabulated in [1]) by: $K_{D_0} = K_D + \pi/4\gamma M^2$, M being the Mach number.

From (40) we obtain, finally,

$$K_{DYH} = \frac{\gamma}{2a^2}\left\{\frac{\bar{u}^2x^2}{2a^2} + \bar{u}^2 - \bar{u}x\left(2\cot\theta - \frac{d}{\gamma-1}\right)\right.$$
$$\left. - 2\bar{u}s_0 - \bar{u}\int_{\theta_w}^{\theta}\frac{z(t-t_s)\,d\theta}{\bar{v}\sin\theta} - \tfrac{1}{2}(x^2+y^2+z^2)\right\} \qquad (43)$$
$$+ \frac{d^2}{4(\gamma-1)^2} - \tfrac{1}{2} - \frac{d\cot\theta + e_0}{\gamma-1},$$

evaluated with $\theta = \theta_s$.

One feature brought to light by the computations [3] deserves comment: though K_{DYH} is always positive (as it should be), $(P_0)_s$ is negative for the smaller Mach numbers.[15] Thus, at Mach numbers up to limits which increase as θ_s decreases, yaw theoretically decreases the axial drag, though not the drag, on the conical head. In this, the present theory agrees qualitatively with the linearized theory, for small θ_s and not too large M, as one would expect.[16]

Observations of conical projectiles in flight, carried out and analyzed at the Ballistic Research Laboratories, Aberdeen, Maryland, show that the increase of *total* drag with yaw is somewhat larger than that predicted by (43).[17] As one would expect yaw to increase the skin friction and base drag also, it seems that here again the theoretical effect is of the right order of magnitude, and that a more precise check would be difficult.

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[15] $(P_0)_s$ is not given in [3], but may easily be derived from the formula $2(P_0/\bar{p})_s = K_{DYH} + (p_0/\bar{p})_s + 1/2 + \gamma(\bar{u}_s/c)^2/(\gamma-1)\,(1-(\bar{u}_s/c)^2)$. If $\theta_s = 15°$, for example, $(P_0)_s$ is negative for $M < 2.2$, and positive for $M > 2.2$; if $\theta_s = 5°$, the corresponding value of M is about 5. It is found that $(p_0)_s$ is always negative.

[16] See [4, pp. 82, 83]. On the linearized theory, the increase in head drag coefficient due to yaw ($= K_{D_0}K_{DYH}\varepsilon^2$) would be approximately $\pi\varepsilon^2/8$. On the present theory, this increase itself increases with M, varying from roughly a third of $\pi\varepsilon^2/8$ (for $M \fallingdotseq 1$) to roughly twice $\pi\varepsilon^2/8$ in the tabulated range.

[17] By a factor of nearly 2, with an uncertainty of about 20%, when $\theta_s = 10°$ and $M \fallingdotseq 2$. The observations lead to no definite conclusion about the sign of $(P_0)_s$.

A HYPERELLIPTIC FUNCTION AS A NON-LINEAR OSCILLATION*

By F. R. Erskine Crossley

1. Oscillatory Motion on an Ellipse. A particle is constrained to move along an elliptical path without friction, under the action of gravity force. The ellipse is in a vertical plane (fig. 1), with major x-axis vertical; the major semi-axis is of unit length, and the eccentricity is e. Its geometric equation is therefore

$$x^2 + [y^2/(1 - e^2)] = 1 \tag{1}$$

If the particle in motion reaches a maximum height $x = h$, the energy relation gives

$$v^2 = 2g(h - x) \qquad -1 < h < 1$$

and when this tangential velocity v is expressed in terms of x, it becomes

$$(1 - e^2x^2)(dx/dt)^2 = 2g(h - x)(1 - x^2). \tag{2}$$

For any initial x in the limits $h \geqq x > -1$, this has obviously a periodic solution. Two special cases, of eccentricity zero and unity, reduce the equation to simpler forms, which allow integration respectively in terms of elliptic and trigonometric functions. The former of course is the common case of simple pendulum motion, for the ellipse of no eccentricity is a circle. But the general case $0 < e < 1$ has a solution only in the hyperelliptic functions (Bibl: 1), which it is the purpose of this paper to present.

2. Differential Equations Soluble. Hyperelliptic functions are a group which solve the general differential equation

$$(dx/dt)^2 = \sum a_n x^n$$

where the polynomial may have an infinite number of terms with real coefficients a. As just previously implied, if the degree of the polynomial is less than four, the equation degenerates, so that a solution may be found in circular or elliptic functions. But where $n > 4$ only a few isolated cases may be so solved. References may be found (Bibl: 2, 3) which suggest a solution by integration in a power series; but the rapidity with which such a series converges, determines the usefulness of this method.

An equation of the type

$$(dx/dt)^2 = \omega^2 \frac{(h \pm x)(1 - x^2)}{(a - x^2)} \tag{3}$$

may be divided out to show a polynomial of infinite degree.

If in equation (2) the substitution $x = -\cos \varphi$ and $h = -\cos \varphi_0$ is made,

* Being part of a dissertation presented by the author to the faculty of the School of Engineering of Yale University in partial fulfillment of the requirements for the degree of Doctor of Engineering.

the following appears:

$$(1 - e^2 \cos^2 \varphi)(d\varphi/dt)^2 = 2\,\omega^2(\cos \varphi - \cos \varphi_0). \tag{4}$$

It happens that this represents the relative angular motion of a simple pendulum, eccentrically mounted and frictionless, rotating in a centrifugal force field. This was the initial investigation of the author. The limits on the parameters in this equation are (for an oscillation)

$$-\pi < \varphi_0 < \pi, \qquad 0 < e < 1 \tag{5}$$

and ω may be a function $\omega(e, \varphi_0)$.

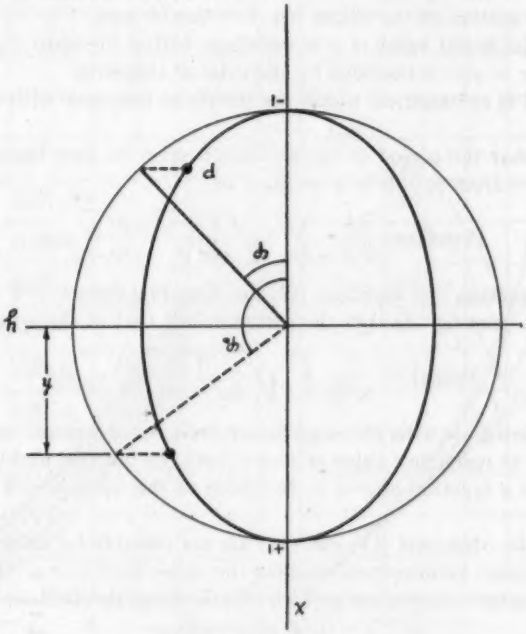

FIG. 1. Ellipse and particle.

The meaning of the substitution $x = -\cos \varphi$ may be seen geometrically in Fig. 1.

Before proceeding with the solution, note that the following equations may be reduced to the form of equation (2) or (4):

$$(1 - 2y - by^2)\dot{y}^2 = c^2(1 - ay)(1 - 2y)y^3 \tag{6}$$

$$\mathrm{dn}^4 u \cdot \dot{u}^2 - \omega^2(a - \mathrm{sn}\, u) = 0 \tag{7}$$

$$\ddot{\varphi} + \omega^2\left[\frac{1 + k \cos \varphi + e^2 \cos^2 \varphi}{(1 - e^2 \cos^2 \varphi)^2}\right] \sin \varphi = 0 \tag{8}$$

$$\ddot{x} + \frac{\omega^2}{2(a - x^2)^2}\,[\pm a + 2b(a - 1)x \pm (3a - 1)x^2 \pm x^4] = 0 \tag{9}$$

where the parameter limits are as in (5) and

$$1 < a < \infty, \qquad 0 < b < 1$$

and k, ω are not limited, except to be real.

There are of course many other forms which can be found by various substitutions; for instance, a $\tan \varphi/2$ substitution can reduce certain equations containing eighth and seventh degree polynomials of the form

$$(P_8)\, \ddot{x} + (P_7)\, \dot{x}^2 = 0.$$

3. Solution of the Equation. Consider the equation (4). Quite apart from the relation to the motion on the ellipse (eq. 1) it can be seen

1. that if the initial value of φ is anywhere within the strip $-\varphi_0 < \varphi < \varphi_0$, the function $\varphi = \varphi(t)$ is bounded by the sides of this strip.

2. that $\varphi(t)$ is symmetrical about the points of tangency with the boundary lines.

It follows that the period of the oscillation must be four times the time required to move from $\varphi = 0$ to $\varphi = \pm\varphi_0$, or

$$\text{Period} = \frac{4}{\sqrt{2}\cdot\omega} \int_0^{\varphi_0} \sqrt{\frac{1 - e^2 \cos^2 \varphi}{\cos \varphi - \cos \varphi_0}}\, d\varphi. \tag{10}$$

With the function x of equation (2), the bounding strip is $-1 < x < h$, and it can be seen from fig. (1) that the period is half that of the φ function, or

$$\text{Period} = \frac{2}{\sqrt{2g}} \int_{-1}^{h} \sqrt{\frac{1 - e^2 x^2}{(h - x)(1 - x^2)}}\, dx. \tag{11}$$

The same integrals with the same lower limit but a general upper limit will give the time to reach any value of φ or x from the bottom, and in this manner define time as a function of φ or x. Inversion of the solution will give $\varphi = \varphi(t)$ or $x = x(t)$.

However, the equations (11) and (12) are not suitable for integration, for the integrand becomes infinite upon reaching the upper limit h or φ_0. This is avoided by using the same substitution as with elliptic integrals (Bibl: 4, 5).

$$\sqrt{2}\, \omega t = \int_0 \sqrt{\frac{1 - e^2 \cos^2 \varphi}{\cos \varphi - \cos \varphi_0}} \cdot d\varphi. \tag{12}$$

Let

$$\sin \varphi/2 = k \sin \zeta \tag{13}$$

where

$$k = \sin \varphi_0/2 \tag{14}$$

$$\omega t = \int_0^{\zeta} \sqrt{\frac{1 - e^2 (1 - 2k^2 \sin^2 \zeta)^2}{1 - k^2 \sin^2 \zeta}} \cdot d\zeta \tag{15}$$

and the solution is defined as the hyperelliptic integral function $L(e, k, \zeta)$ having the two moduli e, k, where k is the modulus of the associated elliptic integral.

Further following the elliptic procedure, the inversion of this function is taken,

$$\omega t = L(e, k, \zeta) \tag{16}$$

$$\zeta = \text{ham } \omega t \tag{17}$$

$$\sin \zeta = \text{sham } \omega t \tag{18}$$

where ζ is the upper limit of the integral; equation (17) is read hyperelliptic amplitude of ωt; equation (18) gives the sine hyperelliptic amplitude. (The h is prefixed because a postfixed h is ordinarily used for hyperbolic, as in sinh or

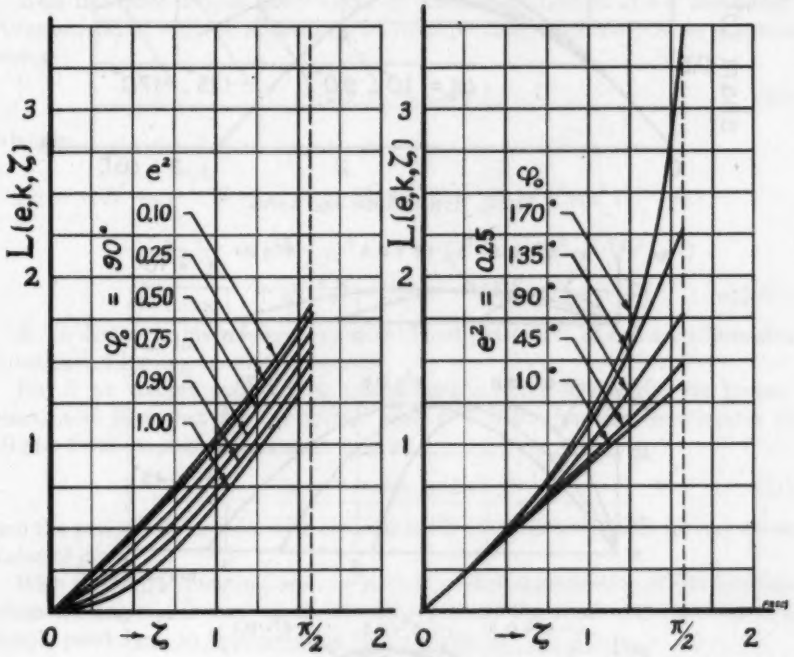

FIG. 2. Values of integral $L(e, k, \zeta)$

cosh, and amh is also used in this sense by the authors of the Smithsonian tables of elliptic functions.)

The solutions of equations (4) and (2) are therefore

$$\begin{aligned} \varphi &= 2 \sin^{-1} (k \text{ sham } \omega t) \\ x &= 2k^2 \text{ sham}^2 \omega t - 1. \end{aligned} \tag{19}$$

Several curves of the functions are given in figs. (2, 3, 4). It may be noted that the ζ-function has an advantage in plotting and comparisons, for it is always within the limits ± 1, not varying in amplitude with k (i.e. φ_0) as does the φ-function.

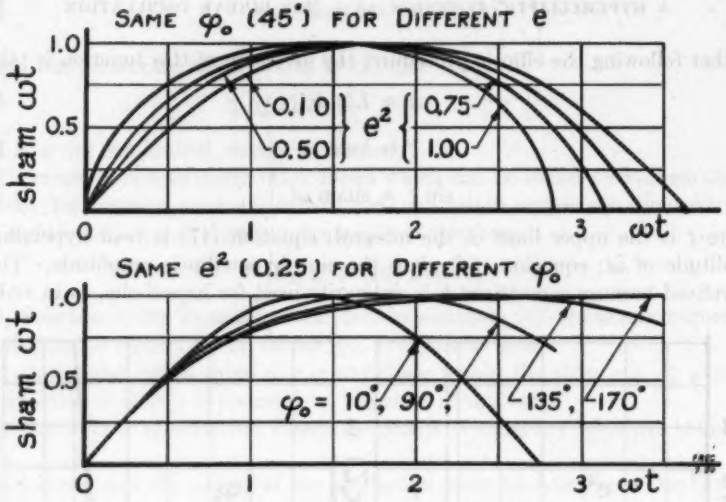

FIG. 3. Hyperelliptic sine waves

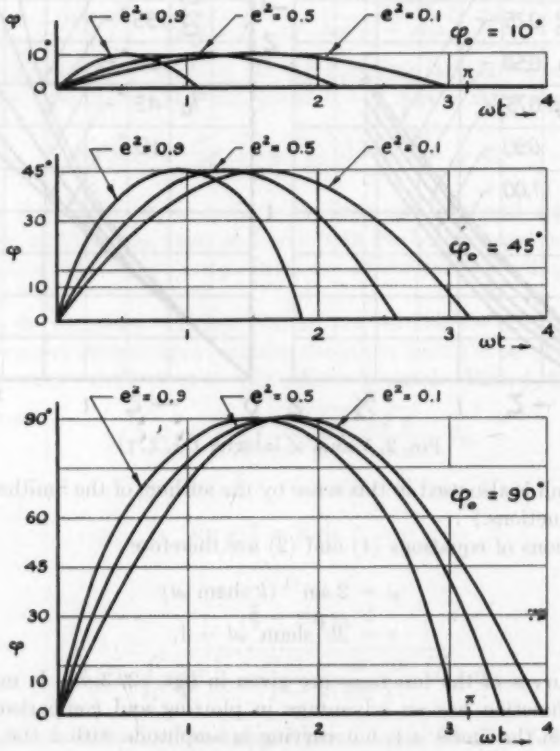

FIG. 4. Pendulum swing (φ) curves

Tables of the function for thirty values of the parameters are given in the appendix. This is thought sufficient, for interpolation is simple—the variation of either parameter causes a smooth change in the value of the function. Simpson's rule was used in the evaluation.

4. Periods of the Oscillation. The period of the oscillation given by equations (10) and (11) are respectively four times, and twice the value of the complete integral L' when divided by ω. This complete integral is the right-hand side of equation (15) when the upper limit is $\pi/2$.

Here the value is most easily found by reading the bottom line of the tables. Alternatively it may be approximated by expanding equation (15) in a power series:

$$\text{Period } T_\varphi = (4/\omega)L' \tag{20}$$

where

$$\begin{aligned} L' &= \int_0^{\pi/2} [1 - e^2 (1 - 2k^2 \sin^2 \zeta)^2]^{1/2} [1 - k^2 \sin^2 \zeta]^{-1/2} \, d\zeta \\ &= \tfrac{1}{2}\pi \left[(1 - \tfrac{1}{2}e^2 - \tfrac{1}{8}e^4 \cdots) + \tfrac{1}{4}k^2 (1 - \tfrac{7}{4}e^2 + \tfrac{15}{8}e^4 \cdots) \right. \\ &\qquad \left. + \tfrac{3}{64}k^4 (3 - \tfrac{15}{2}e^2 - \tfrac{165}{8}e^4 \cdots) + \cdots\right] \qquad \text{approx.} \end{aligned}$$

E. G. Keller in his reference to these functions (Bibl: 2) has an alternative substitution leading to a similar series.

But if we confine ourselves to comparing the effect of changing the parameters, note first that for the special case $e = 0$ the function deteriorates to elliptic form, so that the solution in φ is

$$\sin \varphi/2 = \sin \varphi_0/2 \operatorname{sn} \omega t \tag{21}$$

and the period will increase with the amplitude φ_0. The same holds for any other value of e.

With the elliptic function case, we have a further degeneration of the function when the angle φ_0 is very small—it will represent the small-angle motion of a simple pendulum, so that equation (21) reduces to

$$\varphi = \varphi_0 \sin \omega t \tag{22}$$

The ratio of the periods of (21) to (22) is well-known to be given by

$$T_\varphi/T_0 = K/(\pi/2) \qquad (e = 0) \tag{23}$$

where T_φ is the period with amplitude φ_0, T_0 is the period with small angle and K is the value of the complete elliptic integral $F(k, \pi/2)$ for the modulus k (eq. 14).

Now if we call the complete hyperelliptic integral L', as already suggested, since the elliptic is only a particular case of the hyperelliptic, we may generalize equation (23) as

$$T_\varphi/T_0 = L'(e, k)/(\pi/2) \tag{24}$$

provided that ω is a constant. If ω is itself a function of the parameters, there is a modification of the above, depending on the relation. Suppose ω is a function of e, k, another parameter a, and a constant:

$$\omega = \omega(e, k, a, c)$$

In equation (22) we get down to the simplest form

$$\omega = \omega(0, \varphi_0/2, a, c) = \omega_0 \qquad \varphi_0 \to 0$$

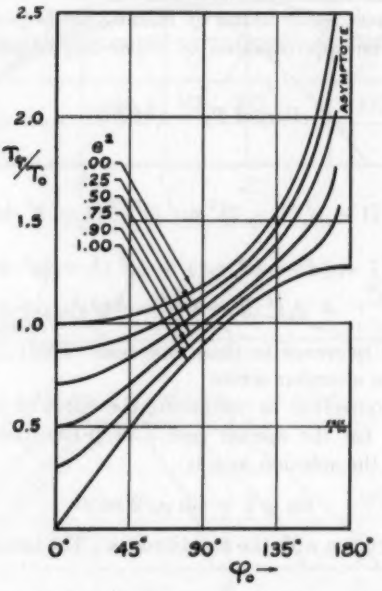

FIG. 5. Period variation from circular function period

In equation (21) the fundamental frequency will be

$$f = (1/2\pi)[(\pi/2K)\omega]$$

so following the same reasoning as before, equation (24) is fully expressed as

$$\frac{T_\varphi}{T_0} = \frac{L'(e, k)}{\pi/2} \cdot \frac{\omega_0}{\omega(e, k, a, c)} \tag{25}$$

Fig. 5 shows the relation of equation (24), using the tabulations for values of L'.

5. Extension of the Parameter Limits. If in equation (2) the parameter h is greater than unity, then the consideration of energy shows that there is still some potential left when the particle reaches the top of the ellipse. It will there-

fore have a velocity at the top and that will convert the oscillation into a continuous varying rotation.

The equation (4) transformed is

$$(1 - e^2 \cos^2 \varphi)\, \dot{\varphi}^2 = 2\, \omega^2 (\cos \varphi + h) \tag{26}$$

and for $h \geqq 1$ it has therefore a solution corresponding closely to the amplitude function, instead of to the sine-amplitude.

As $h \to \infty$, the function $\varphi(t)$ approaches the elliptic of the second kind.

$$\begin{aligned} &(1 - e^2 \cos^2 \varphi)\, \dot{\varphi}^2 = 2h\, \omega^2 \\ &E(e, \varphi + \pi/2) = \sqrt{2h} \cdot \omega t \quad \text{for} \quad h \to \infty \end{aligned} \tag{27}$$

The x-function is not so changed, for again considering energy, no matter with what velocity the particle reaches the top, at that point $\dot{x} = 0$, as x can only vary between ± 1. Hence the x-function is always an oscillation having shorter periods as h increases.

For all $h > 1$ the parameter k does not enter. No calculations of the functions of this type are presented.

6. Representation on the Poincare Phase-plane. The phase-plane diagram (Bibl: 6, 7) is used for plotting the velocity against the displacement of oscillations; for a simple harmonic vibration it appears as an ellipse. This can be seen from the energy equation

$$\tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{2} k x^2 = \text{const}$$

or

$$y^2 + \omega^2 x^2 = c \qquad \text{for } y = \dot{x}$$

which is an equation of an ellipse.

Three diagrams, corresponding to the values of $e^2 = 0,\ 0.5,\ 0.75$, are shown in fig. (6) for equations (4) and (26). The borderline case, $h = 1$ or $\varphi_0 = \pi$ forms the separatrix. The curves of $e = 0$ are of course the elliptic motion, the others the hyperelliptic. Note that the scale of the ordinate varies between the diagrams.

The center of the vortices, $y = \dot{\varphi} = 0$ and $\varphi = 0,\ \pm 2\pi$ etc. show points of stable equilibrium about which the oscillations occur. The cross of the saddles, $y = \dot{\varphi} = 0$ and $\varphi = \pm\pi,\ \pm 3\pi$ etc. show points of unstable equilibrium. The lines for $h > 1$ never reach $\dot{\varphi} = 0$, and are seen to include large second-harmonic terms as e gets large.

7. Conclusion. These hyperelliptic functions appear then to be a larger class, of which elliptic and circular functions are a degenerate form; they themselves are members of the still larger group of Abelian functions. (Bibl: 8–14). Two other examples of their occurrence may perhaps be mentioned: the equation of motion of a top spinning on a smooth table is given by Webster (Bibl: 15), and secondly the Schwartzian transformation is often in the same differential form.

Any numerical integration method is not as accurate as could be desired. An

attempt was made to find another way, but unsuccessfully: in case someone should be interested, the approach still seems promising. It is to find some transformation similar to Landen's transformation used in elliptic function evaluation (Bibl: 16, 17), or by use of Gauss's arithmetico-geometrical mean

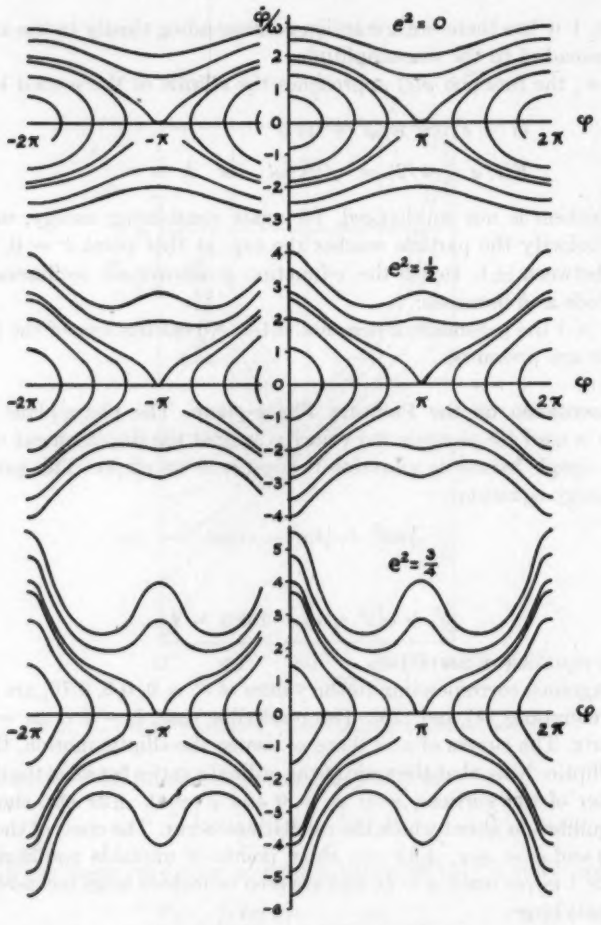

FIG. 6. Phase-plane representation*

series, to move the parameter e in value progressively nearer unity or zero. For just as the elliptic integral curve is bracketted between the straight line and the log tangent curve, so this hyperelliptic L function is bracketted between the elliptic F function and the curve $2k(1 - \cos \zeta)$ corresponding to $e = 0$ and $e = 1$.

* In each case the lines shown are those for $\varphi_0 = \pi/4, \pi/2, 3\pi/4$, the separatrix π or $h = 1$, and $h = 2$ and 3.

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APPENDIX

HYPERELLIPTIC FUNCTIONS

TABLE I

$\varphi_0 = 10°$, $k = 0.08716$

$L(\epsilon, k, \zeta)$ or ωt						ham ωt ζ rad	$\zeta°$	sham ωt sin ζ	$\varphi°$
$\epsilon^2 = 0.10$	0.25	0.50	0.75	0.90	1.0				
0.1656	0.1512	0.1234	0.0873	0.0553	0.0027	0.1745	10	0.1736	1°44′24″
0.3312	0.3024	0.2470	0.1749	0.1110	0.0105	0.3491	20	0.3420	3°25′ 0″
0.4970	0.4538	0.3708	0.2629	0.1676	0.0234	0.5236	30	0.5000	4°59′42″
0.6628	0.6054	0.4950	0.3516	0.2253	0.0408	0.6981	40	0.6428	6°25′24″
0.8289	0.7572	0.6196	0.4410	0.2842	0.0623	0.8727	50	0.7660	7°39′24″
0.9951	0.9093	0.7446	0.5311	0.3444	0.0872	1.0472	60	0.8660	8°39′28″
1.1614	1.0615	0.8699	0.6218	0.4056	0.1147	1.2217	70	0.9397	9°23′44″
1.3278	1.2139	0.9955	0.7130	0.4676	0.1440	1.3963	80	0.9848	9°50′52″
1.4943	1.3664	1.1212	0.8045	0.5300	0.1743	1.5708	90	1.0000	10° 0′ 0″

Note: In each table the last line gives L' for $L(\epsilon, k, \zeta)$.

TABLE II

$\varphi_0 = 45°$, $k = 0.38268$

$L(e, k, \zeta)$ or αt						ham αt ζ rad	$\zeta°$	sham αt sin ζ	$\varphi°$
$e^2 = 0.10$	0.25	0.50	0.75	0.90	1.0				
0.1658	0.1514	0.1239	0.0881	0.0567	0.0116	0.1745	10	0.1736	7°37′14″
0.3325	0.3043	0.2504	0.1809	0.1213	0.0462	0.3491	20	0.3420	15° 2′30″
0.5013	0.4601	0.3817	0.2820	0.1988	0.1025	0.5236	30	0.5000	22° 3′44″
0.6728	0.6197	0.5193	0.3934	0.2909	0.1790	0.6981	40	0.6428	28°28′48″
0.8473	0.7837	0.6639	0.5155	0.3972	0.2734	0.8727	50	0.7660	34° 5′36″
1.0251	0.9520	0.8151	0.6474	0.5161	0.3827	1.0472	60	0.8660	38°42′34″
1.2057	1.1241	0.9720	0.7875	0.6450	0.5036	1.2217	70	0.9397	42° 9′ 8″
1.3885	1.2990	1.1331	0.9334	0.7809	0.6325	1.3963	80	0.9848	44°16′48″
1.5724	1.4755	1.2963	1.0822	0.9204	0.7654	1.5708	90	1.0000	45° 0′ 0″

TABLE III

$\varphi_0 = 90°$, $k = 0.70711$

$L(e, k, \zeta)$ or αt						ham αt ζ rad	$\zeta°$	sham αt sin ζ	$\varphi°$
$e^2 = 0.10$	0.25	0.50	0.75	0.90	1.0				
0.1662	0.1520	0.1250	0.0900	0.0600	0.0212	0.1745	10	0.1736	14°41′40″
0.3360	0.3093	0.2586	0.1950	0.1429	0.0853	0.3491	20	0.3420	27°59′28″
0.5127	0.4762	0.4077	0.3237	0.2575	0.1895	0.5236	30	0.5000	41°24′36″
0.6994	0.6552	0.5758	0.4790	0.4047	0.3309	0.6981	40	0.6428	54° 4′ 6″
0.8985	0.8504	0.7643	0.6606	0.5819	0.5052	0.8727	50	0.7660	65°35′44″
1.1114	1.0615	0.9724	0.8656	0.7851	0.7071	1.0472	60	0.8660	75°31′20″
1.3383	1.2879	1.1978	1.0900	1.0089	0.9305	1.2217	70	0.9397	83°16′56″
1.5770	1.5265	1.4362	1.3283	1.2471	1.1686	1.3963	80	0.9848	88°16′20″
1.8226	1.7720	1.6818	1.5739	1.4927	1.4142	1.5708	90	1.0000	90° 0′ 0″

TABLE IV

$\varphi_0 = 135°$, $k = 0.92388$

$L(e, k, \zeta)$ or μt						ham αt ζ rad	$\zeta°$	sham αt sin ζ	$\varphi°$
$e^2 = 0.10$	0.25	0.50	0.75	0.90	1.0				
0.1666	0.1527	0.1260	0.0919	0.0630	0.0281	0.1745	10	0.1736	18°27′50″
0.3394	0.3142	0.2666	0.2077	0.1608	0.1114	0.3491	20	0.3420	36°50′26″
0.5246	0.4922	0.4400	0.3592	0.3033	0.2476	0.5236	30	0.5000	55° 1′30″
0.7288	0.6933	0.6438	0.5496	0.4903	0.4323	0.6981	40	0.6428	72°51′46″
0.9595	0.9236	0.8732	0.7781	0.7184	0.6600	0.8727	50	0.7660	90° 6′ 6″
1.227	1.190	1.139	1.043	0.9826	0.9239	1.0472	60	0.8660	106°16′46″
1.544	1.504	1.445	1.342	1.277	1.216	1.2217	70	0.9397	120°29′32″
1.924	1.873	1.796	1.673	1.597	1.527	1.3963	80	0.9848	130°58′ 4″
2.356	2.289	2.183	2.028	1.932	1.848	1.5708	90	1.0000	135° 0′ 0″

TABLE V

$\varphi_0 = 170°$, $k = 0.99620$

$L(\varepsilon, k, \zeta)$ or ωt						ham ωt ζ rad	$\zeta°$	sham ωt sin ζ	$\varphi°$
$\varepsilon^2 = 0.10$	0.25	0.50	0.75	0.90	1.0				
0.1668	0.1529	0.1264	0.0927	0.0642	0.0303	0.1745	10	0.1736	19°55′23″
0.3408	0.3161	0.2696	0.2124	0.1672	0.1202	0.3491	20	0.3420	39°50′29″
0.5295	0.4985	0.4411	0.3721	0.3193	0.2669	0.5236	30	0.5000	59°44′55″
0.7414	0.7083	0.6474	0.5749	0.5199	0.4661	0.6981	40	0.6428	79°38′ 5″
0.9881	0.9548	0.8936	0.8208	0.7656	0.7117	0.8727	50	0.7660	99°28′48″
1.290	1.254	1.188	1.110	1.052	0.9962	1.0472	60	0.8660	119°14′56″
1.692	1.643	1.554	1.452	1.378	1.311	1.2217	70	0.9397	138°49′ 7″
2.328	2.240	2.079	1.893	1.760	1.646	1.3963	80	0.9848	157°39′45″
3.689	3.491	3.121	2.671	2.334	1.992	1.5708	90	1.0000	170° 0′ 0″

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ON APPROXIMATE EXPRESSIONS FOR THE EXPONENTIAL INTEGRAL AND THE ERROR FUNCTION*

By Richard Bellman

1. Introduction. In many important applications of mathematical physics, the final numerical answer depends upon the evaluation of a Laplace transform

$$g(y) = \int_0^\infty e^{-xy} f(x)\, dx, \qquad \text{Re } (y) > 0. \tag{1}$$

Integrating by parts n times, the result is

$$g(y) = \frac{f(0)}{y} + \frac{f^1(0)}{y^2} + \cdots + \frac{f^{(n-1)}(0)}{y^n} + \frac{1}{y^n}\int_0^\infty e^{-xy} f^{(n)}(x)\, dx. \tag{2}$$

If $f(0) \neq 0$, we may use the first n terms as an approximation to $g(y)$. The error term is the integral on the right-hand side of (2) which is bounded by

$$y^{-(n+1)} \operatorname*{Max}_{0 \leq x < \infty} | f^{(n)}(x) | \tag{3}$$

for y real and by

$$| y |^{-n} \operatorname*{Max}_{0 \leq x < \infty} | f^{(n)}(x) | / \text{Re } (y), \tag{4}$$

in case y is complex. In many cases of importance, the series

$$S(y) = \sum_{k=0}^{\infty} f^{(k)}(0)/y^{k+1} \tag{5}$$

is divergent for all y. Consequently, in order to make maximum use of (2), n must be chosen as a function of y to minimize the expressions in (3) or (4). The fact that $S(y)$ is a divergent asymptotic series as $y \to \infty$ imposes a lower bound on the magnitude of the error term.

The question arises then as to whether it is possible to obtain a better approximation to $g(y)$ than that given by (2) using some rational function of $1/y$, with coefficients determined by the $f^{(k)}(0)$. One way of obtaining those rational approximations is that of expanding $S(y)$ into a formal continued fraction and using the n-th convergents.

The purpose of the present note is to indicate another method which under all circumstances yields an error term less than or equal to that given by (3) or (4), and in general may be expected to improve it considerably. After indicating the general method, we apply it to the two most commonly encountered non-elementary transcendents

$$E_i(y) = \int_y^\infty \frac{e^{-x}}{x}\, dx = e^{-y}\int_0^\infty \frac{e^{-xy}}{1+x}\, dx$$

$$1 - \text{Erf } (y) = \frac{1}{\sqrt{2\pi}}\int_y^\infty e^{-x^2}\, dx = \frac{e^{-\sqrt{v}}e^{-1/4}}{\sqrt{2\pi}}\int_0^\infty \frac{e^{-x\sqrt{v}}}{(1+x)^{1/2}}\, dx. \tag{6}$$

* The above results were obtained in the course of research sponsored by the Rand Corporation.

The following result is obtained:

THEOREM: *If $y > 0$, there exist for each $n \geq 1$, polynomials in $1/y$, $Q_n(y)$, $R_n(y)$ with the properties that*

$$\left| \int_0^\infty \frac{e^{-xy}}{1+x}\,dx - \frac{R_n(y)}{Q_n(y)} \right| \leq \frac{n!}{2^{2n+1}y^{n+1}Q_n(y)} < \frac{n!}{2^{2n+1}y^{n+1}}, \tag{7}$$

where

$$\text{(a)} \qquad Q_n(y) = 1 + \sum_{i=1}^{n} a_i y^{-1},$$

$$\text{(b)} \qquad a_k = n!\,\frac{2^{n-k+1}}{[(n+1-k)!]^2} \prod_{l=1}^{n-k+1} \left(\frac{(n+1)^2 - l^2}{2l+1} \right), \tag{8}$$

$$\text{(c)} \qquad R_n(y) = P_n(y) + \sum_{i=1}^{n} a_i y^{-i} P_{n-i}(y) - a_{n+1} y^{-n-1},$$

$$\text{(d)} \qquad P_k(y) = \sum_{l=0}^{k-1} (-1)^l l! y^{-(l+1)}, \qquad k = 1, 2, \cdots.$$

If y is complex, the right side of (7) is replaced by

$$\frac{n!}{2^{2n+1} \mid y \mid^n \mid Q_n(y) \mid \operatorname{Re}(y)}. \tag{9}$$

A similar result holds for any integral of the form

$$I_\alpha(y) = \int_0^\infty \frac{e^{-xy}}{(1+x)^\alpha}\,dx. \tag{10}$$

Applied to the integral in (7), $I_1(y)$, the method discussed in (2) et seq. yields the bound $n!/y^{n+1}$. The best possible bound is obtained by taking $n = [y]$, yielding an error term approximately equal to $e^{-y}\sqrt{2\pi y}$. The best possible bound obtained from the above theorem is obtained by taking $n = 2y$ and is approximately $e^{-4y}\sqrt{2\pi y}$.

Although these functions in the cases $\alpha = \frac{1}{2}$ and 1 have been extensively tabulated for real y, there may still be some practical application of the inequalities we obtain, since in several important applications these functions occur with complex argument. We discuss below a device which may be employed if Re (y) is small.

2. The method. Let us consider first the simplest case where $f(0) \neq 0$, y is real, and we will be satisfied with an approximation with an error of order $1/y^2$ as $y \to \infty$. Integrating by parts once, we obtain

$$\begin{aligned} g(y) &= \frac{f(0)}{y} + \frac{1}{y}\int_0^\infty e^{-xy} f^1(x)\,dx \\ &= \frac{f(0)}{y} + \frac{1}{y}\int_0^\infty e^{-xy}[f^1(x) + af(x) + b]\,dx - \frac{a}{y}\int_0^\infty e^{-xy} f(x)\,dx - \frac{b}{y^2}. \end{aligned} \tag{1}$$

From this follows

$$g(y)\left(1+\frac{a}{y}\right)-\left(\frac{f(0)}{y}+\frac{b}{y^2}\right)=\frac{1}{y}\int_0^\infty e^{-xy}[f^1(x)+af(x)+b]\,dx, \tag{2}$$

whence

$$\left|g(y)\left(1+\frac{a}{y}\right)-\left(\frac{f(0)}{y}+\frac{b}{y^2}\right)\right|\leq \operatorname*{Max}_{0\leq x<\infty}|f^1(x)+af(x)+b|/y^2. \tag{3}$$

The parameters a and b are now to be chosen so as to minimize the right-hand side. In general, this is a difficult problem, which we do not propose to discuss here. Nevertheless, in many cases, a few trials will yield values of a and b which appreciably reduce the error term.

Let us now proceed to the general case. After n-integrations by parts, we have

$$g(y)=P_n(y)+\frac{1}{y^n}\int_0^\infty e^{-xy}f^{(n)}(x)\,dx, \tag{4}$$

where we have set

$$P_n(y)=\sum_{k=0}^{n-1}f^{(k)}(0)y^{-(k+1)},\qquad n=1,2,\cdots. \tag{5}$$

We now write $g(y)$ as follows:

$$\begin{aligned} g(y) &= P_n(y) \\ &+ 1/y^n\int_0^\infty e^{-xy}[f^{(n)}(x)-a_1f^{(n-1)}(x)+\cdots+a_nf(x)+a_{n+1}]\,dx \\ &- 1/y^n\sum_{i=1}^n a_i\int_0^\infty e^{-xy}f^{(n-i)}(x)\,dx-a_{n+1}/y^{n+1}. \end{aligned} \tag{6}$$

Utilizing (4) to eliminate the integrals $\int_0^\infty e^{-xy}f^{(n-i)}(x)\,dx$, this leads to the inequality

$$\left|g(y)-\frac{R_n(y)}{Q_n(y)}\right|\leq\frac{\operatorname*{Max}_{0\leq x<\infty}|f^{(n)}(x)+a_1f^{(n-1)}(x)+\cdots+a_nf(x)+a_{n+1}|}{y^{n+1}|Q_n(y)|}, \tag{7}$$

where we have set

$$\begin{aligned} Q_n(y) &= 1+\sum_{i=1}^n a_iy^{-i} \\ R_n(y) &= P_n(y)+\sum_{i=1}^n a_iy^{-i}P_{n-i}(y)-a_{n+1}y^{-n-1}. \end{aligned} \tag{8}$$

The error term depends upon the polynomial $Q_n(y)$ and the functional

$$(9)\quad c_n(f) = \operatorname*{Min}_{a_i} \operatorname*{Max}_{0\le x<\infty} |f^{(n)}(x) + a_1 f^{(n-1)}(x) + \cdots + a_n f(x) + a_{n+1}|.$$

We may conceive of $c_n(f)$ as a measure of the deviation of $f(x)$ from a function of the form

$$(10)\quad h(x) = \sum_{k=1}^{N} p_k(x) e^{b_k x},$$

where each $p_k(x)$ is a polynomial in x.

In the succeeding section we determine $c_n(f)$ for the case where $f(x) = 1/1 + x$.

3. Application to the Exponential Integral. Referring to (6) of §1, we see that it is sufficient to treat the Laplace transform

$$(1)\quad g(y) = \int_0^\infty \frac{e^{-xy}}{1+x}\,dx.$$

Consequently, we wish to determine

$$(2)\quad \operatorname*{Min}_{a_i} \operatorname*{Max}_{0\le x<\infty} \left| \frac{n!}{(1+x)^{n+1}} - \frac{a_1(n-1)!}{(1+x)^n} + \cdots + \frac{(-1)^n a_n}{1+x} + (-1)^n a_{n+1} \right|.$$

Upon setting $\dfrac{1}{1+x} = \dfrac{1+x'}{2}$, this reduces to

$$(3)\quad \operatorname*{Min}_{a_i} \operatorname*{Max}_{-1\le x\le 1} \left| \frac{n!\,(1+x)^{n+1}}{2^{n+1}} - \frac{(n-1)!\,a_1(1+x)^n}{2^n} + \cdots + (-1)^n a_{n+1} \right|.$$

Since the degree of the polynomial is $(n+1)$ and the coefficient of the highest degree term is $n!/2^{n+1}$, it follows that the minimal polynomial is the $(n+1)$st Chebychev polynomial

$$(4)\quad \begin{aligned} \frac{n!\,T_{n+1}(x)}{2^{n+1}} &= \frac{n!\cos(n+1)\arccos x}{2^{2n+1}} \\ &= \frac{n!\,(1+x)^{n+1}}{2^{n+1}} - \frac{(n-1)!\,a_1(1+x)^n}{2^n} + \cdots + (-1)^n a_{n+1}, \end{aligned}$$

and that

$$(5)\quad c_n(1/(1+x)) = n!/2^{2n+1}.$$

Referring to (7) of the previous section, we see that before we can apply the inequality with confidence, we must know something about the location of the roots of $Q_n(y) = 0$. Using (4), we shall determine the a_i and show that they are all positive, whence $Q_n(y) > 1$ for $y > 0$.

For $n = 1, 2, 3$, the roots of $Q_n(y) = 0$ are all negative, and we hazard a guess that this is true in general. This result, if valid, would be of importance in applying the inequality to the case where y is complex.

Since

$$T_{n+1}(x) = T_{n+1}(-1 + (1 + x)) = \sum_{k=0}^{n+1} \frac{T_{n+1}^{(k)}(-1)(1 + x)^k}{k!}, \tag{6}$$

we obtain from the identity of (4),

$$\frac{(-1)^{n-k+1} a_{n-k+1} k!}{2^{n-k+1}} = \frac{n!}{2^{n+1}} \frac{T_{n+1}^{(k)}(-1)}{k!}. \tag{7}$$

To determine $T_{n+1}^{(k)}(-1)$, we use the fact that $T_{n+1}(x)$ satisfies the differential equation

$$(1 - x^2)T''_{n+1}(x) - xT'_{n+1}(x) + (n + 1)^2 T_{n+1}(x) = 0. \tag{8}$$

Hence $T_{n+1}^{(k)}(x)$ satisfies the equation

$$(1 - x^2)T_{n+1}^{(k+2)}(x) - (2k + 1)xT_{n+1}^{(k+1)}(x) + ((n + 1)^2 - k^2)T_{n+1}^{(k)}(x) = 0. \tag{9}$$

Thus

$$T_{n+1}^{(k+1)}(-1) = -\left[\frac{(n + 1)^2 - k^2}{2k + 1}\right] T_{n+1}^{(k)}(-1), \tag{10}$$

whence finally

$$\begin{aligned} T_{n+1}^{(k)}(-1) &= (-1)^{n+k+1} \prod_{l=1}^{k} \left(\frac{(n + 1)^2 - l^2}{2l + 1}\right), && k = 1, 2, \cdots \\ &= (-1)^{n+1}, && k = 0. \end{aligned} \tag{11}$$

Substituting in (7), we obtain

$$\begin{aligned} a_k &= \frac{n!\, 2^{n+1-k}}{[(n + 1 - k)!]^2} \prod_{l=1}^{n+1-k} \left(\frac{(n + 1)^2 - l^2}{2l + 1}\right), && k \neq n + 1, \\ &= n!, && k = n + 1. \end{aligned} \tag{12}$$

This completes the proof of the theorem stated above.

If Re (y) is small, the term $1/\text{Re}\,(y)$ may greatly increase the error term, particularly for small n. We may overcome this to some extent by writing

$$\begin{aligned} &\int_0^\infty e^{-xy} f^{(n)}(x)\, dx \\ &= \frac{1}{y^n} \int_0^\infty \frac{e^{-x}}{(1 + x)^2} \left[\frac{(-1)^n n!}{(1 + x)^{n-2}} + \frac{a_1(n - 1)!}{(1 + x)^{n-3}} + \cdots + (-1)^{n-1} a_{n-1}\right] \end{aligned} \tag{13}$$

minus the appropriate terms.

From this we obtain as a bound

$$\operatorname*{Min}_{a_i} \operatorname*{Max}_{0 \leq x < \infty} \left| \frac{(-1)^n n!}{(1 + x)^{n-1}} + \cdots + (-1)^{n-1} a_{n-1} \right| \Big/ |y|^n = \frac{n!}{2^{2n-1} |y|^n}, \tag{14}$$

which, if Re (y) is small, will be less than $n!/2^{2n+1}|y|^n$ Re (y).

4. Application to the Error Function. Referring to (6) of §1, it is sufficient to consider

$$g(y) = \int_0^\infty \frac{e^{-xy}}{(1+x)^{1/2}}\,dx. \tag{1}$$

In this case, we have the problem of determining

$$\operatorname*{Min}_{a_i}\ \operatorname*{Max}_{0\le x<\infty} |f^{(n)}(x) + a_1 f^{(n-1)}(x) + \cdots + a_n f(x) + a_{n+1}|, \tag{2}$$

where $f(x) = (1+x)^{-1/2}$. Taking $a_{n+1} = 0$, we obtain as a bound

$$\operatorname*{Min}_{a_i}\ \operatorname*{Max}_{0\le x<\infty} \left| (1+x)^{-1/2}\left(\frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\cdots\left(\frac{2n-1}{2}\right)}{(1+x)^n} + \cdots + a_n\right)\right| \tag{3}$$

$$\le \operatorname*{Min}_{a_i}\ \operatorname*{Max}_{0\le x<\infty} \left| \frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{2^n(1+x)^n} + \cdots + a_n\right| = \frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{2^n 2^{n-1}}.$$

Similarly, we can obtain bounds for the general integral

$$I_\alpha(y) = \int_0^\infty \frac{e^{-xy}}{(1+x)^\alpha}\,dx, \tag{4}$$

or integrals of the form

$$J_\alpha(y) = \int_0^\infty \frac{e^{-xy-x^n}}{(1+x)^\alpha}\,dx. \tag{5}$$

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ON THE NUMERICAL EVALUATION OF FREDHOLM INTEGRAL EQUATIONS WITH THE AID OF THE LIOUVILLE-NEUMANN SERIES

By Carl Wagner

We consider the numerical evaluation of a Fredholm integral equation of second kind

$$\varphi(x) = f(x) + \lambda \int_a^b K(x, z)\, \varphi(z)\, dz \tag{1}$$

where a, b and λ are given constants, $f(x)$ and $K(x, z)$ are given functions, and $\varphi(x)$ is to be determined. The solution may be found by successive approximations $\psi^{(0)}(x), \cdots \psi^{(n)}(x)$. Replacing $\varphi(x)$ in Eq. (1) by $\psi^{(n)}(x)$, we may calculate the difference of the left-hand and the right-hand member as

$$\Delta^{(n)}(x) = \psi^{(n)}(x) - \lambda \int K(x, z)\, \psi^{(n)}(z)\, dz - f(x). \tag{2}$$

Here and in the following equations the limits of integration, a and b, are omitted for the sake of brevity. Let $\epsilon^{(n)}(x)$ be the deviation of the n^{th} approximation $\psi^{(n)}(x)$ from the true solution $\varphi(x)$,

$$\epsilon^{(n)}(x) = \psi^{(n)}(x) - \varphi(x). \tag{3}$$

Upon substituting Eq. (3) in Eq. (2) and using Eq. (1), it follows that

$$\epsilon^{(n)}(x) - \lambda \int K(x, z)\, \epsilon^{(n)}(z)\, dz = \Delta^{(n)}(x) \tag{4}$$

where $\Delta^{(n)}(x)$ for a given approximate solution $\psi^{(n)}(x)$ is known according to Eq. (2). If it were possible to solve Eq. (4) directly for $\epsilon^{(n)}(x)$, the true solution would be obtained from Eq. (2) at once. A solution of Eq. (4), however, encounters the same difficulties as the solution of the original Eq. (1). Thus one has to use an approximation.

The Liouville-Neumann series is obtained if the integral in Eq. (4) is disregarded. Then one has $\Delta^{(n)}(x)$ as an approximate value of $\epsilon^{(n)}(x)$, which is to be subtracted from $\psi^{(n)}(x)$ in order to obtain $\psi^{(n+1)}(x)$. Thus the $(n + 1)^{\text{th}}$ term of the Liouville-Neumann series equals

$$k^{(n+1)}(x) = \psi^{(n+1)}(x) - \psi^{(n)}(x) = -\Delta^{(n)}(x). \tag{5}$$

If $K(x, z)$ as a function of z does not vary excessively, one may replace $K(x, z)$ in Eq. (4) by its average value in order to obtain an approximation. Thus, since $\int dz = \int dx$,

$$\epsilon^{(n)}(x) - \lambda \left[\int \epsilon^{(n)}(x)\, dx\right] \left[\int K(x, z)\, dz\right] \Big/ \int dx \cong \Delta^{(n)}(x). \tag{6}$$

Upon integrating Eq. (6) with respect to x and regrouping, it follows that

$$\int \epsilon^{(n)}(x)\,dx \cong \frac{\int \Delta^{(n)}(x)\,dx}{1 - \lambda\left\{\int\left[\int K(x, z)\,dz\right]dx\right\}\Big/\int dx.} \tag{7}$$

Substituting Eq. (7) in Eq. (6) and solving for $\epsilon^{(n)}(x)$, we obtain

$$\epsilon^{(n)}(x) \cong \Delta^{(n)}(x) + \frac{\lambda\left[\int \Delta^{(n)}(x)\,dx\right]\left[\int K(x, z)\,dz\right]\Big/\int dx}{1 - \lambda\left\{\int\left[\int K(x, z)\,dz\right]dx\right\}\Big/\int dx}. \tag{8}$$

In particular, it seems profitable to use the negative value of the right-hand member in Eq. (8) instead of $-\Delta^{(n)}(x)$ for the last term of the Liouville-Neumann series used in a computation. Replacing $-\Delta^{(n)}(x)$ in Eq. (8) by $k^{(n+1)}(x)$ according to Eq. (5), one has

$$\varphi(x) \cong f(x) + k^{(1)}(x) + \cdots + k^{(n+1)}(x) + \chi^{(n+1)}(x) \tag{9}$$

where the $k^{(n)}(x)$ values are the terms of the conventional Liouville-Neumann series, and

$$\chi^{(n+1)}(x) = \frac{\left[\int k^{(n+1)}(x)\,dx\right]\left[\int K(x, z)\,dz\right]\Big/\int dx}{1 - \lambda\left\{\int\left[\int K(x, z)\,dz\right]dx\right\}\Big/\int dx} \tag{10}$$

represents a last correction term.

This procedure may be regarded as a special application of a method for solving a set of linear equations suggested by Gauss in his lectures and reported by Dedekind [1]. The usefulness of this method has been pointed out recently by Zurmuehl [2].

For the sake of illustration, we calculate solutions of integral equation (1) with

$$f(x) = x^4 - 2x^3 + x \tag{11a}$$

$$K(x, z) = \begin{cases} z(1 - x) \text{ for } 0 \leqq z \leqq x \\ x(1 - z) \text{ for } x \leqq z \leqq 1 \end{cases} \tag{11b}$$

and the interval of integration extends from 0 to 1. With these special functions, Eq. (1) is the dimensionless form of an equation used by Wiarda [3] for calculating the deflection of a horizontal beam compressed in axial direction with a uni-

formly distributed load normal to the axis and with hinged ends. This problem may also be stated in the form of a differential equation whose solution is [4]

$$\varphi(x) = \frac{24}{\lambda^2}\left\{\frac{\cos\left[\frac{1}{2}\lambda^{\frac{1}{2}}(1-2x)\right]}{\cos\left(\frac{1}{2}\lambda^{\frac{1}{2}}\right)} - 1\right\} - \frac{12x(1-x)}{\lambda} \tag{12}$$

with $\lambda_1 = \pi^2$ as the first eigenvalue.

Numerical values of approximations obtained with and without the term $\chi^{(n+1)}(x)$ are shown in Table I for various values of λ and $x = \frac{1}{2}$. Approximations including the term $\chi^{(n+1)}(x)$ are nearly as good as or even better than solutions involving a subsequent term of the Liouville-Neumann series.

TABLE I

Approximate Values of $\varphi(x = \frac{1}{2})$ Defined by the Integral Equation

$$\varphi(x) = (x^4 - 2x^3 + x) + \lambda\int K(x, z)\varphi(z)\,dz$$

Where $K(x, z)$ is Defined by Equation (11b)

λ	2.4674	3.5531	4.8361
λ/λ_1	0.25	0.36	0.49
$f(\frac{1}{2}) + k^{(1)}(\frac{1}{2})$	0.3909	0.4254	0.4661
$f(\frac{1}{2}) + k^{(1)}(\frac{1}{2}) + \chi^{(1)}(\frac{1}{2})$	0.4103	0.4708	0.5653
$f(\frac{1}{2}) + k^{(1)}(\frac{1}{2}) + k^{(2)}(\frac{1}{2})$	0.4105	0.4660	0.5415
$f(\frac{1}{2}) + k^{(1)}(\frac{1}{2}) + k^{(2)}(\frac{1}{2}) + \chi^{(2)}(\frac{1}{2})$	0.4153	0.4824	0.5900
$\varphi(\frac{1}{2})$ from Eq. (12)	0.4170	0.4889	0.6138

In the case of a numerical integration of the values of $k^{(n)}(x)$, the calculation of the expression $\chi^{(n+1)}(x)$ requires less labor than that of a subsequent term of the Liouville-Neumann series, since in Eq. (10) only the factor $\int K(x, z)\,dz$ is a function of x. Use of Eqs. (9) and (10) may therefore save labor if $K(x, z)$ as a function of z does not vary excessively. But this method should not be used if $K(x, z)$ has a high peak for $x = z$. In the latter case another approach is more helpful [5].

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ROTORS IN SPHERICAL POLYGONS

By Michael Goldberg

1. Introduction. In an earlier paper by the author [1], the term *rotor* was applied to convex curves which may be rotated in polygons through every orientation while keeping contact with each side of the polygon. The term may be extended to three-dimensional shapes. For example, surfaces of constant width may be considered as rotors in equilateral parallelepipeds. Rotors for the regular tetrahedron and the regular octahedron have been derived by Ernst Meissner [2].

A circular cone can be inscribed in a regular pyramid. Furthermore, this cone, when rotating on its axis, will keep continual contact with the faces of the pyramid. It will be shown that there are other conical shapes which can be inscribed and rotated in regular pyramids while keeping contact with the lateral faces of the pyramid. Such cones can be derived by considering the equivalent convex spherical curves in which they cut a sphere constructed about the vertex as a center. These curves are rotors in the spherical polygons which correspond to the regular pyramids.

The case of the curve of constant width on the sphere has already been considered by Blaschke [3] and Santaló [4]. This curve is inscribed in a lune bounded by two meridians. Since the width is measured by the arc of the common normal great circle joining the midpoints of the meridians, the curve of constant width must always touch the meridians at these midpoints. In the plane, a curve of constant width is not constrained against sliding along the parallel lines which touch it. Therefore, another pair of parallel lines may be added thus making it a rotor in a rhombus. However, the spherical curve of constant width is constrained against sliding. The instantaneous center of rotation must lie on the common normal great circle. If, at the same time, the curve were tangent to another fixed lune with the same width, the instantaneous center of rotation would have to lie on its common normal also. Therefore, the instantaneous center would then be the intersection of these normals, the same fixed point for all orientations. Hence, it follows that this curve cannot be a rotor in a spherical quadrilateral except in the trivial case in which it is a circular rotor.

However, non-circular rotors in spherical quadrilaterals are still possible although they are not curves of constant width. Furthermore, it will be shown that rotors can be constructed in regular spherical polygons of any number of sides and for any length of the arcs. As the length of the arcs approach zero, the cases of the plane rotors are approached. As the lengths of the arcs increase and the polygon approaches a single great circle, the rotor approaches the same great circle.

2. A kinematic construction of a rotor in a spherical quadrilateral. Inscribe an equilateral spherical triangle ABC in the given quadrilateral so that point A lies at the midpoint of the top boundary. (See Fig. 1.) Let this triangle be carried

by the rotor which is about to be constructed. Let D be the point opposite to A on the quadrilateral. Take the points A, B, C, D as points on the contour of the rotor being constructed. Then as the point A of the rotor moves to the right along the quadrilateral, let the point B of the rotor slide upward along the left edge.

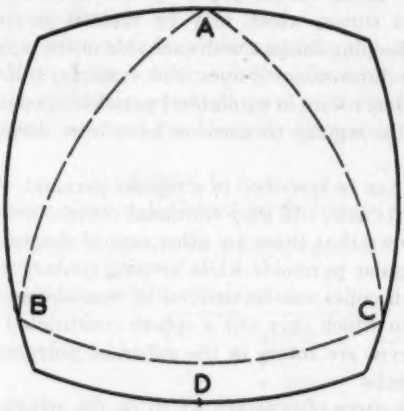

FIG. 1. Equilateral spherical triangle inscribed in regular spherical quadrilateral

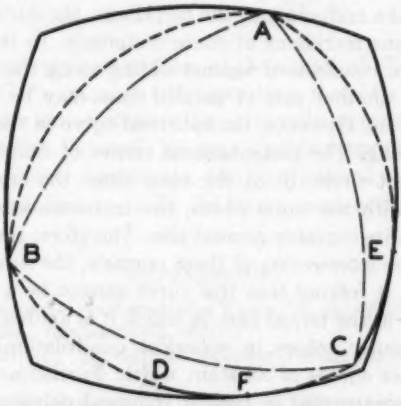

FIG. 2. Generation of rotor in spherical quadrilateral

Then the right edge of the quadrilateral molds a boundary curve for the rotor beginning at C. Continue this motion until the triangle is symmetric about the diagonal of the quadrilateral. (See Fig. 2.) Let the contact points of the rotor with the quadrilateral now be E on the right edge and F on the bottom edge. The arcs of the rotor from C to E and from D to F have been molded. In other words, the traced positions of the right edge and of the bottom edge of the fixed

quadrilateral on the moving plane of the rotor will have the arcs CE and DF as envelopes. Complete the rotor by constructing arcs similar to CE and DF, that is, CF equal to CE and DC equal to DB. The three boundary arcs AB, BC and CA will then be equal. Because of the symmetry of the equilateral rotor, the contacts will continue for the complete rotation of the rotor.

3. A construction of a rotor in the odd-sided spherical polygons. Given a spherical polygon of an odd number of sides (see Fig. 3), construct an inscribed doubly symmetric quadrilateral $ABCD$ which lies symmetrically about an alti-

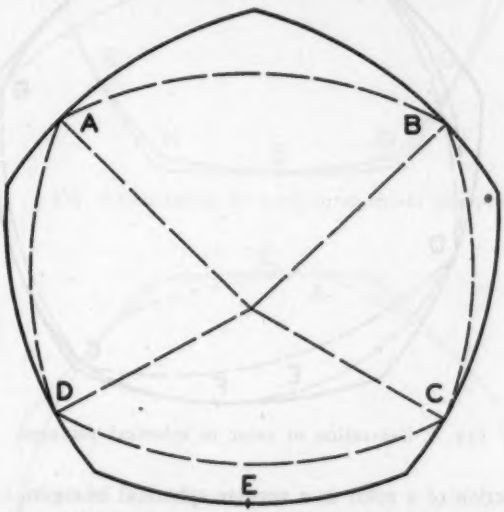

FIG. 3. Doubly symmetric spherical quadrilateral inscribed in regular spherical pentagon

tude so that the points A and D lie on sides adjoining the vertex of the polygon farthest to the left and the points B and C lie on the sides adjoining the vertex farthest to the right, and so that the normals to the polygon at the points A, B, C, D intersect at a single point. The latter point is the instantaneous center of rotation of the rotor carrying the points A, B, C, D as the rotor begins its motion. Let E be the midpoint of the bottom edge of the polygon. Take the points A, B, C, D, E as points on the contour of the rotor to be constructed. Rotate the rotor clockwise so that the points A and D move along the edges of the polygon until the quadrilateral $ABCD$ is symmetric about another altitude of the given polygon. (See Fig. 4.) If the point F is now the contact point of the rotor with the bottom edge of the polygon, the arc EF of the rotor will have been molded by this bottom edge. In a similar fashion, each edge of the polygon, except the ones on which the points A and D were sliding, will have molded an arc of the boundary of the rotor. In Fig. 4, these additional arcs are CG and BH. Complete the rotor by constructing arcs similar to these while

preserving the double symmetry of the quadrilateral $ABCD$. Then the boundary arc AHB will be equal to the boundary arc DFC, while the boundary arc AD will be equal to the boundary arc BC.

If the given polygon is a spherical triangle, the quadrilateral $ABCD$ degenerates to an arc AB which is equal to the altitude of the given spherical triangle.

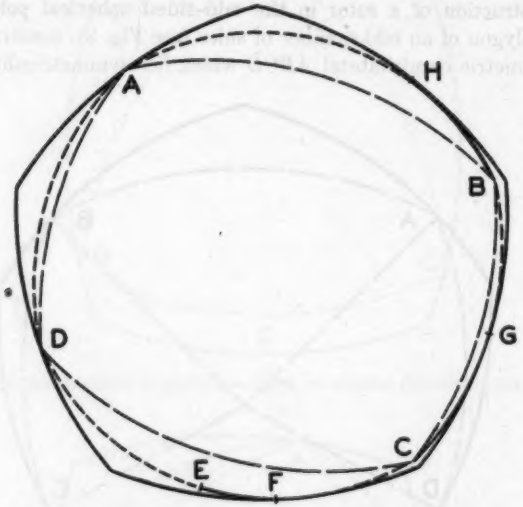

FIG. 4. Generation of rotor in spherical pentagon

4. A construction of a rotor in a regular spherical hexagon. Given a regular spherical hexagon, let A, B, C, D, E, F, the midpoints of the sides, be points on the boundary of the rotor about to be constructed. (See Fig. 5.) Let the rotor turn clockwise so that the points A and B move along the edges of the polygon. Among the boundary arcs molded by the edges of the polygon is the arc DG which is molded by the bottom edge. If the rotor is turned counter-clockwise so that A and F move along edges of the polygon, the arc CG is molded. The point G is determined by the intersection of these two arcs. The point H is determined in a corresponding way. The solid arcs, shown in Fig. 5, are molded by these motions.

With the points A and B moving along the edges of the polygon, turn the rotor clockwise until the point G touches the bottom edge. Continue the rotation, but now with B and G moving along edges. During the latter motion, the dotted portion of CB is molded by an edge of the hexagon while the arc AF is molded by the top edge. A symmetrical counter-clockwise rotation will mold the other two dotted arcs which complete the boundary of the rotor.

It should be noted that the five points A, B, F, G, H are sharp points of the rotor; that is, there is a discontinuity in the slope of the tangent at these points.

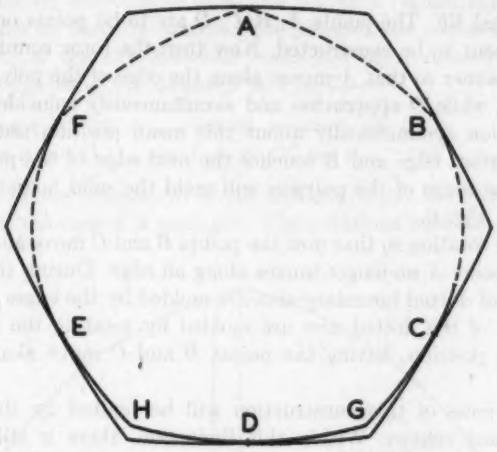

FIG. 5 Generation of rotor in spherical hexagon

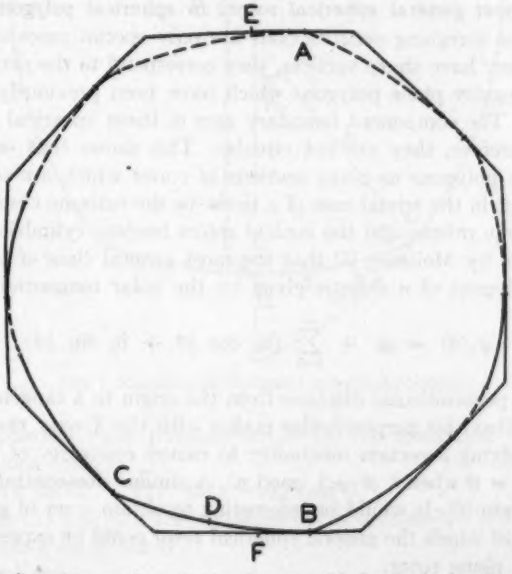

FIG. 6. Generation of rotor in even-sided spherical polygon

5. A construction of a rotor in the even-sided spherical polygons. Given a regular spherical polygon of any even number of sides, inscribe an isosceles triangle ABC (see Fig. 6) so that A and B lie to the right of the centers E and F of a pair of opposite sides. Let AD be the bisector of the vertex angle CAB

and let AD equal EF. The points A, B, C, D are to be points on the boundary of the rotor about to be constructed. Now turn the rotor counter-clockwise in an arbitrary manner so that A moves along the edge of the polygon until it coincides with E, while D approaches and simultaneously coincides with F. Continue this motion symmetrically about this mean position until the point C touches the bottom edge and B touches the next edge of the polygon. During this motion, the edges of the polygon will mold the solid boundary arcs of the rotor shown in Fig. 6.

Continue the rotation so that now the points B and C move along edges of the polygon. The point A no longer moves along an edge. During this motion, half of the number of dotted boundary arcs are molded by the edges of the polygon. The other half of the dotted arcs are molded by rotating the rotor clockwise from its initial position, letting the points B and C move along edges of the polygon.

The arbitrariness of this construction will be limited by the necessity for making the rotor convex. Within this limitation, there is still a continuous infinity of different shapes.

6. On the most general spherical rotors in spherical polygons. The rotors described in the foregoing constructions are only special cases of the possible rotors. Since they have sharp vertices, they correspond to the extreme circular-arc rotors in regular plane polygons which have been previously described by the author [1]. The component boundary arcs of these spherical rotors are not plane and, therefore, they are not circular. This shows that one cannot use rotors of plane polygons as plane sections of cones which are to be rotors in pyramids except in the trivial case of a circle or the extreme cases in which the pyramids become prisms and the conical rotors become cylindrical rotors.

It was shown by Meissner [5] that the most general class of plane rotors in the regular polygons of n sides is given by the polar tangential equation

$$p_n(\theta) = a_0 + \sum_{k=1}^{\infty} (a_k \cos k\theta + b_k \sin k\theta)$$

where p is the perpendicular distance from the origin to a tangent to the rotor, θ is the angle that this perpendicular makes with the X-axis, the a_k and b_k are constants satisfying a certain inequality to ensure convexity of the rotor, and $a_k = 0$ and $b_k = 0$ when $k \not\equiv \pm 1 \pmod{n}$. A similar demonstration was given later by Fujiwara [6]. It would be interesting to obtain a set of geometric variables in terms of which the general spherical rotor could be expressed as simply as the general plane rotor.

7. Calculation of the principal dimensions of the extreme rotors. Of all the possible rotors in the regular polygons, the rotors described in Sections 2 to 5 may be considered as *extreme* cases since they are characterized by sharp vertices. The principal dimensions of these rotors are, therefore, chosen to be the lengths of the arcs between these vertices and the distances of the midpoints

of the rotor boundary curves from these arcs. Given a regular spherical polygon of n arcs of length $2a$, then the angles of the polygon, the radius of the inscribed circle, the radius of the circumscribed circle and other related quantities are derived as follows, using the notation of Fig. 7.

$$\sin C = (\cos \pi/n)/\cos a, \qquad \tan m = \sin a/\cot C$$
$$\tan b = \tan a/\cos C, \qquad \cos J = \sin 2C \cos a, \qquad \tan u = -\sin a/\cot 2C$$

The calculation of the dimensions of the extreme rotors in these polygons will be illustrated for the case of a pentagon. The notations used are shown in Figs. 7, 8 and 9.

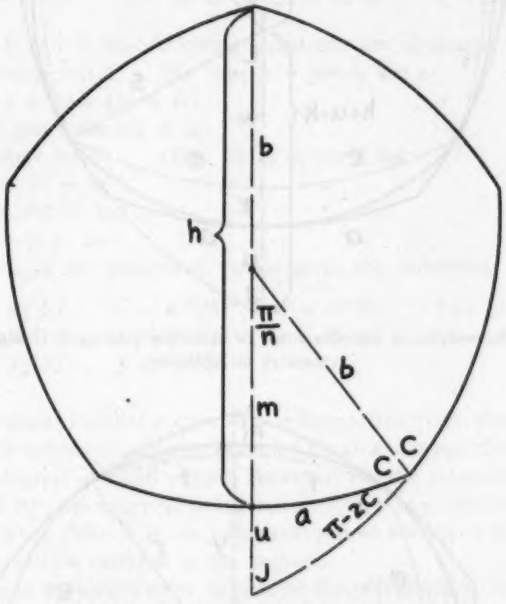

FIG. 7 Parameters of regular spherical polygon

From the spherical right triangles involving the associated quantities, the following relations hold.

(1) $\sin t = \sin k \sin C$, (2) $\sin e = \sin t \sin G$

From equations (1) and (2), the equation (3) is obtained.

(3) $\sin e = \sin k \sin C \sin G$

Similarly, we have

(4) $\sin s = \sin (h + u - k) \sin J$

(5) $\cot L = \cos (h + u - k) \tan J$

(6) $\sin e = \sin s \sin L$, (7) $\cot G = \cos k/\cot C$

Equation (3) gives the quantity $\sin e$ as a function of the unknown quantity k and known quantities derivable from the given polygon. Similarly, equation

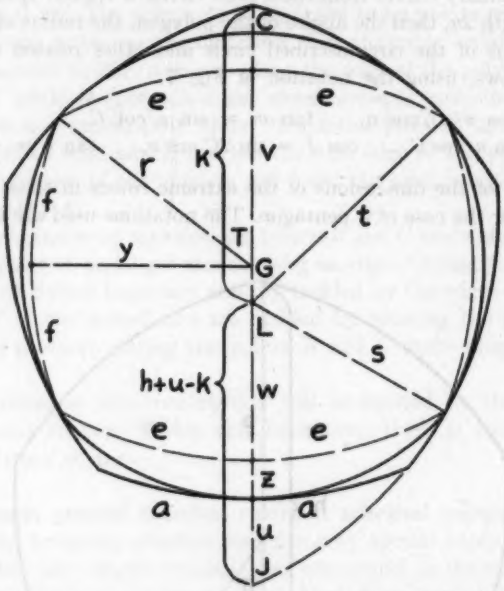

FIG. 8 Dimensions of extreme rotor in spherical pentagon (Long axis of symmetry on altitude)

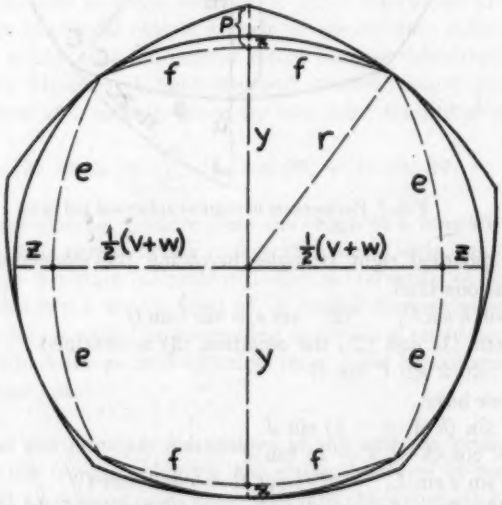

FIG. 9 Dimensions of extreme rotor in spherical pentagon (Short axis of symmetry on altitude)

(6) gives sin e as another function of k and the known quantities. Therefore, by successive trials for k, until the same value for sin e is obtained in equations (3) and (6), one may approximate k and the derived quantities.

For example, if the regular spherical pentagon is composed of 45° arcs, then

$$a = 22\tfrac{1}{2}^\circ \qquad C = 61^\circ 07' \qquad u = 31^\circ 15' \qquad J = 38^\circ 37'$$
$$m = 34^\circ 45' \qquad b = 40^\circ 37' \qquad h = 75^\circ 22'$$

The value for k converges on the value 52°04′, from which the following values are obtained.

$$G = 41^\circ 54' \qquad s = 30^\circ 33' \qquad L = 65^\circ 09' \qquad e = 27^\circ 28'$$

From Figs. 8 and 9, the following equations are obtained.

(8) $\tan v = \cos G \tan t$, (9) $\tan w = \cos L \tan s$

(10) $\cos r = \cos e \cos \frac{1}{2}(v + w)$

(11) $\tan T = \tan e / \sin \frac{1}{2}(v + w)$

(12) $\sin f = \sin r \cos T$, (13) $\tan y = \sin T \tan r$

(14) $z = h - k - w$

(15) $\sin p = \cot C \tan f$

(16) $x = h - p - 2y$

The substitution of the numerical values gives the following.

$$t = 43^\circ 40' \qquad v = 35^\circ 23' \qquad w = 13^\circ 56' \qquad T = 51^\circ 15'$$
$$r = 36^\circ 15' \qquad f = 21^\circ 44' \qquad y = 29^\circ 46' \qquad z = 9^\circ 22'$$
$$p = 12^\circ 42' \qquad x = 3^\circ 08'$$

8. Parallel rotors. Parallel curves to a spherical rotor are also rotors. These are not rotors in spherical polygons bounded by arcs of great circles but, rather, in spherical polygons of small circles. However, for the internal parallel curve at a distance of 90°, the spherical polygon for which it is a rotor is again made of arcs of great circles since it is the polar polygon of the given polygon. In this case, the rotor will be external to the polygon.

Another way of making a rotor outside of the polygon is to hold an internal rotor fixed while the polygon is rotated around it. The vertices of the polygon will all trace the same shape which can serve as a rotor around the fixed polygon. The 90° parallel to this shape, when it is a simple convex curve, will be an internal rotor of the polar polygon.

9. Applications. A plane rotor or a solid rotor taken with its associated stator is a movable mechanism although seemingly overconstrained. Such a pair of elements is called a *higher* pair by engineers and machine designers. These higher pairs may be included with exceptional linkages and other mechanisms to form a class which may be called *singular mechanisms*. The class of spherical rotors discussed here are another contribution to the class of singular mechanisms.

The plane rotors have found occasional application as cam devices. In the same way, the spherical rotors may find possible application as special cams.

These singular cam mechanisms possess the advantage over the usual cam mechanisms in that they do not require the cam follower to be held against the cam by a spring. Vibration difficulties are reduced.

Another application is a negative one. Where cones of circular section are desired, these non-circular cones may be obtained through various sources of error in production. If the test for circularity consists of multiple contacts with the faces of a regular pyramid, these non-circular cones will pass the test. Therefore, the test must be modified or augmented by other means to reduce this hazard.

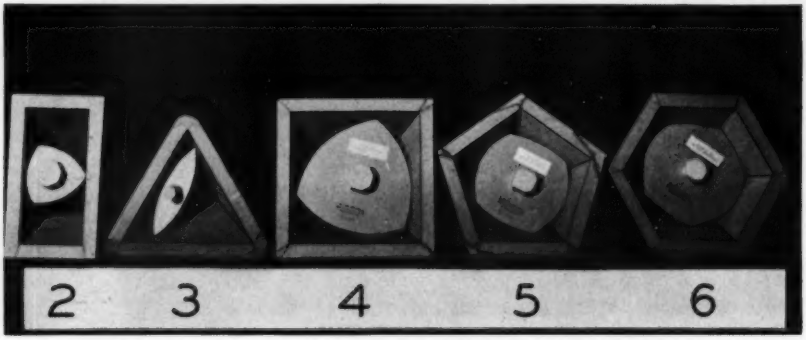

FIG. 10 Photograph of several conical rotors in regular pyramids

The models of the conical rotors in pyramids, shown in Fig. 10, are built upon the plane sections of the cones passing through the center of the sphere and the boundaries of the spherical rotors. These plane sections are not plane rotors in plane polygons. The axes of the cones oscillate with respect to the pyramids during the rotation, making the plane sections cut the pyramids in varying irregular polygons.

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ON THE CONVERGENCE OF UNSTABLE APPROXIMATE SOLUTIONS OF THE HEAT EQUATION TO THE EXACT SOLUTION

By Werner Leutert

Introduction. In a recent paper by O'Brien, Hyman and Kaplan [1] it is shown that a certain parabolic difference approximation does not possess numerically stable solutions whenever the mesh ratio $r = \Delta t/(\Delta x)^2$ exceeds $\frac{1}{2}$.

The question of whether it is possible to find solutions of the difference equation which converge to the exact solution of the differential equation for any positive r or whether instability implies lack of convergence will be answered in this paper. It will be shown that for any positive value of r the difference equation admits infinitely many exact solutions which converge to the exact solution of the differential equation.

1. The Problem.[1] The function $u(x, t)$ satisfies the differential equation

$$\partial u/\partial t = \partial^2 u/\partial x^2 \qquad t > 0 \qquad 0 < x < 1$$

the boundary conditions $u(0, t) = u(1, t) = 0$, $t \geqq 0$ and the initial condition

$$u(x, 0) = f(x) \qquad 0 < x < 1. \tag{1}$$

The exact solution D of this problem is well known and given by

$$u(x, t) = \sum_{n=1}^{\infty} a_n \sin \pi n x \, e^{-\pi^2 n^2 t} \tag{2}$$

where

$$a_n = 2 \int_0^1 f(x) \sin \pi n x \, dx.$$

It will be assumed that $f(x)$ is sectionally continuous in the interval $[0, 1]$ and that it has one sided derivatives at all interior points of that interval. These conditions are sufficient to show that $u(x, t)$ is the unique solution of the problem.

The special case $f(x) \equiv 1$ is the simple parabolic problem O'Brien, Hyman and Kaplan refer to in their paper.

2. The difference equation. Two difference equations are used in reference one to obtain numerical results. For the difference equation used by Richardson (page 232) the authors point out themselves (page 238) that there exist convergent exact solutions for any positive r, while all numerical solutions, obtained by making use of the difference equation, are unstable. The author of this paper has proved this fact (reference two). The second case will be considered here.

[1] The notation is slightly different from the one used by O'Brien, Hyman and Kaplan.

The interval $0 \leqq x \leqq 1$ is divided into M[2] subintervals of equal length Δx. The mesh ratio r is defined by the relation

$$(\Delta x)^2 r = \Delta t.$$

Let $v(x, t)$ be an exact solution Δ of the difference equation

$$v(x, t + \Delta t) - v(x, t) = r[v(x + \Delta x, t) + v(x - \Delta x, t) - 2v(x, t)] \tag{3}$$

(equation (2) in reference one). By separation of variables it can easily be found that

$$v(x, t) = \sum_{n=1}^{\infty} b_n \sin \pi n x \left[1 - 4r \sin^2 \frac{\pi n}{2M}\right]^{tM^2/r} \tag{4}$$

is a formal solution of (3), each term of which satisfies (3) and the same boundary conditions as $u(x, t)$, namely

$$v(0, t) = v(1, t) = 0 \qquad t \geqq 0.$$

Therefore each term of (4) is an eigenfunction of the difference equation. The b_n are arbitrary constants.

3. The exact solutions. While it is necessary to known the value of $u(x, 0)$ in (1) for almost all values of x in the interval $0 < x < 1$, it is sufficient to know the values of $v(x, 0)$ on the meshpoints of the first line in order to start a numerical computation.

The convergence theorem of section five applies to any one of the infinitely many exact solutions of (3) which will be denoted by $v_{pM}(x, t)$. They may be found in the following way.

To any preassigned positive integer p and any fixed $r \geqq \frac{1}{4}$ there exists a positive constant M_0 such that for any integral $M > M_0$ a positive integer $k(p, M)$ may be uniquely determined so, that

$$1 - 4r \sin^2 (n + p)\pi/2M \geqq 0 \qquad n = 1, 2, \cdots, k(p, M)$$

and

$$1 - 4r \sin^2 (k + 1 + p)\pi/2M \leqq 0.$$

For $0 < r \leqq \frac{1}{4}$ we choose $k(p, M) = M - p$. It is obvious that for fixed p and r

$$\lim_{M \to \infty} k(p, M) = \infty. \tag{5}$$

If we put in (4) $b_n = 0$ for all $n > k(p, M)$ and choose the remaining k different $b_n(p, M)$ in such a way that to any preassigned $\epsilon > 0$ there exists an $M_1(\epsilon, p)$ such that for any $M > M_1$

$$|b_n(p, M) - a_n| < \epsilon \quad \text{uniformly for} \quad n = 1, 2, \cdots, k \tag{6}$$

[2] Throughout this paper M always denotes a positive integer.

then for any $M > \max(M_0, M_1)$ the exact solution $v_{pM}(x, t)$ is defined by the formula

$$v_{pM}(x, t) = \sum_{n=1}^{k} b_n(p, M) \sin \pi n x [1 - 4r \sin^2 (\pi n/2M)]^{tM^2/r} \tag{7}$$

in other words whenever $r \geqq \frac{1}{4}$ the summation on n in (7) extends to within p of the stage when the bracketed quantity becomes negative, where p is an arbitrarily fixed positive integer.

4. Special cases of initial values. The initial condition (1) for D corresponds to the initial condition

$$v_{pM}(x, 0) = \sum_{n=1}^{k} b_n \sin \pi n x \tag{8}$$

taken on the meshpoints

$$Mx = q \qquad q = 1, 2, \cdots, M - 1$$

of the first line.

Case 1. For

$$b_n = a_n = 2 \int_0^1 f(x) \sin \pi n x \, dx \tag{9}$$

condition (6) is certainly satisfied. In this case the initial values for $v_{pM}(x, t)$ are chosen in such a way that the integral

$$\int_0^1 [f(x) - v_{pM}(x, 0)]^2 \, dx$$

is minimized. (Least square error approximation).

Case 2. For the special case $f(x) \equiv 1$, $r \leqq \frac{1}{4}$ and $p = 1$ we obtain $k = M - 1$ and

$$v_{1M}(x, t) = \sum_{n=1}^{M-1} b_n \sin \pi n x [1 - 4r \sin^2 (\pi n/2M)]^{tM^2/r}. \tag{10}$$

Choosing as initial condition $v_{1M}(x, 0) = 1$ on the $M - 1$ meshpoints of the first line $q = 1, 2, \cdots, M - 1$, a simple computation gives

$$\begin{aligned} b_n(M) &= 0, \qquad n \text{ even} \\ b_n(M) &= (2/M) \cot (\pi n/2M) \qquad n \text{ odd.} \end{aligned} \tag{11}$$

Since in this special case

$$a_n = 0, \quad n \text{ even}; \qquad a_n = 4(\pi n)^{-1}, \quad n \text{ odd}$$

Condition (6) follows directly from the fact that

$$h(n) = a_n - b_n = (4/\pi n)[1 - (\pi n/2M) \cot (\pi n/2M)]$$

considered as a function of n, is non negative and possesses a continuous, non negative derivative with respect to n in the interval $1 \leqq n \leqq M$. Therefore (6) is satisfied for

$$M_1 > 4\,(\pi\epsilon)^{-1}$$

and the convergence theorem of section five applies to this case. (10) together with (11) is nothing else but the exact solution of the difference equation given by O'Brien, Hyman and Kaplan on page 235 of reference one (formula 17). The authors assume that this particular exact solution of the difference equation (3) converges for all $r < \frac{1}{2}$, but they offer no proof (page 237).

The proof given here applies only for $r \leqq \frac{1}{4}$. A recently completed thesis by O'Brien (reference three) contains a proof for the same range of r ($r \leqq \frac{1}{4}$). It is different from the special case of the convergence theorem in section five which holds for this particular exact solution.

5. The convergence theorem. *Assumptions:*

1) $t \geqq t_0 > 0$, $r > 0$, $0 \leqq x \leqq 1$. r and t_0 are fixed positive constants.

2) $\sum_{n=1}^{\infty} a_n \sin \pi n x \equiv f(x)$, $|f(x)| < P$, P independent of x

$$a_n = 2\int_0^1 f(x) \sin \pi n x\, dx$$

3) To any $\epsilon > 0$ there exists an $M_1(\epsilon, p)$ such that for any $M > M_1$ $|b_n(p, M) - a_n| < \epsilon$ for $1 \leqq n \leqq k(p, M)$, provided $\lim_{M\to\infty} k = \infty$

4) $1 - 4r \sin^2 (\pi n/2M) \geqq 0$ for $1 \leqq n \leqq k(p, M)$.

THEOREM:

$$\lim_{M\to\infty} v_{pM}(x, t) = u(x, t)$$

or

$$\lim_{M\to\infty} \sum_{n=1}^{k} b_n \sin \pi n x\, [1 - 4r \sin^2 (\pi n/2M)]^{tM^2/r} = \sum_{n=1}^{\infty} a_n \sin \pi n x\, e^{-\pi^2 n^2 t}$$

x, r are fixed. The limit is uniform in t. For $b_n = a_n$ the value of t may be chosen zero, i.e. convergence is proved for $t = 0$. (This is not important for applications because the value of $f(x) = u(x, 0)$ is known.)

6. A Lemma. *Lemma. To any $\epsilon > 0$ there exists an $M_2(\epsilon)$ such that for any $M > M_2$ any fixed $t \geqq 0$ and fixed $r > 0$*

$$|e^{-4tM^2z^2} - (1 - 4r \sin^2 z)^{tM^2/r}| < \epsilon/M$$

uniformly in z for $0 \leqq z \leqq z_0$ where $z_0 = \pi/2$ for $r \leqq \frac{1}{4}$ and $z_0 = \operatorname{Arcsin} (4r)^{-1/2}$ for $r \geqq \frac{1}{4}$.

If $t \geqq t_0 > 0$ then $M_2(\epsilon, t)$ may be chosen equal to $M_2(\epsilon, t_0)$.

Proof. Since the lemma is correct for $t = 0$, assume $t \geqq t_0 > 0$.

a) Let $z_1 = \frac{1}{4}\pi$ for $r \leqq \frac{1}{4}$ and $z_1 = \operatorname{Arcsin} (8r)^{-1/2}$ for $r \geqq \frac{1}{4}$. Then $0 < z_1 \leqq \frac{1}{4}\pi$.

In the interval $z_1 \leqq z \leqq z_0$

$$e^{-4tM^2z^2} \leqq e^{-4t_0M^2z_1^2}$$

and

$$(1 - 4r \sin^2 z)^{tM^2/r} \leqq (1 - 4r \sin^2 z_1)^{t_0M^2/r} \leqq 2^{-t_0M^2/r}. \tag{12}$$

It is obvious that there exists an $M_3(\epsilon, t_0)$ such that for any $t \geqq t_0$ and any $M > M_3$ the lemma is correct for the chosen interval of z.

b) It will be shown that

$$(1 - 4r \sin^2 z)^{tM^2/r} - e^{-4tM^2z^2} < \epsilon/M \tag{13}$$

for $0 \leqq z \leqq z_1$ and the remaining assumptions made in the lemma. (13) is equivalent to

$$F(z) = e^{4tM^2z^2}[(1 - 4r \sin^2 z)^{tM^2/r} - (\epsilon/M)] < 1. \tag{14}$$

The first derivative

$$F'(z) = 4tM^2e^{4tM^2z^2}(1 - 4r \sin^2 z)^{(tM^2/r)-1} G(z) \tag{15}$$

will certainly be non positive for all z in the given interval if

$$G(z) = 2z(1 - 4r \sin^2 z) - \frac{2\epsilon z}{M(1 - 4r \sin^2 z)^{(tM^2/r)-1}} - \sin 2z \tag{16}$$

is non positive. Since

$$G'(z) \leqq \sin^2 z\left[4 - 8r - \frac{2\epsilon}{M \sin^2 z(1 - 4r \sin^2 z)^{(tM^2/r)-1}}\right] \tag{17}$$

and $H(z) = M \sin^2 z[1 - 4r \sin^2 z]^{(tM^2/r)-1} \leqq M^{-1/3}$ for $0 \leqq z \leqq M^{-2/3}$

$$H(z) \leqq M(1 - 4r \sin^2 z_1)^{-1}\left[\left(1 - \frac{16r}{\pi^2 M^{4/3}}\right)^{\pi^2M^{4/3}/16r}\right]^{16t_0M^{2/3}/\pi^2}$$

it is easily seen that there exists a positive $M_4(\epsilon, t_0)$ independent of z such that for any $M > M_4$ and any $t \geqq t_0$

$$H(z) < \epsilon/(2 - 4r) \quad r < \tfrac{1}{2} \tag{18}$$

and

$$H(z) < \epsilon/4r \tag{19}$$

(18) implies that $G'(z)$ in (17) is non positive and since in (16) $G(0) = 0$, it follows that in (15) $F'(z)$ is non positive. (14) implies $F(0) < 1$ and therefore (13) is proved.

c) It will be shown that

$$e^{-4tM^2z^2} - (1 - 4r \sin^2 z)^{tM^2/r} < \epsilon/M \tag{20}$$

for $0 \leqq z \leqq z_1$ and the remaining assumptions made in the lemma. (20) is equivalent to

$$k(z) = 4tM^2z^2 + \ln[(\epsilon/M) + (1 - 4r \sin^2 z)^{tM^2/r}] \geqq 0. \tag{21}$$

Since $K(0) > 0$ and

$$K'(z) = 8tM^2 z\left[1 - \frac{\sin 2z}{2z}\,\frac{1}{1 - 4r\sin^2 z + (\epsilon/M)(1 - 4r\sin^2 z)^{1-(tM^2/r)}}\right] \tag{22}$$

is certainly non negative for

$$(\epsilon/M)(1 - 4r\sin^2 z)^{1-(tM^2/r)} \geqq 4r\sin^2 z. \tag{23}$$

(20) is proved because (23) is an immediate consequence of (19). The lemma at the beginning of this section is therefore proved for $M_2 = \max(M_3, M_4)$.

7. Proof of the convergence theorem. It can easily be verified that any $v_{pM}(x, t)$ given by (7) satisfies all assumptions made in the convergence theorem. From assumption two and Abel's test it follows that to any given $\epsilon > 0$ there exists an $M_5(\epsilon, t_0)$ such that for any $M > M_5$ and any $t \geqq t_0$

$$\left|\sum_{n=k+1}^{\infty} a_n \sin \pi nx\, e^{-\pi^2 n^2 t}\right| < \epsilon. \tag{24}$$

By substituting $z = \pi n/2M$ into the lemma it follows immediately[3] for any $M > \max(M_0, M_1, M_2, M_5)$

$$\left|\sum_{n=1}^{k} b_n \sin \pi nx\,[1 - 4r\sin^2 \pi n/2M]^{tM^2/r} - \sum_{n=1}^{\infty} a_n \sin \pi nx\, e^{-\pi^2 n^2 t}\right|$$

$$= \left|\sum_{n=1}^{k} (b_n - a_n) \sin \pi nx\, e^{-\pi^2 n^2 t}\right.$$

$$+ \sum_{n=1}^{k} (b_n - a_n + a_n) \sin \pi nx\,[(1 - 4r\sin^2(\pi n/2M))^{tM^2/r} - e^{-\pi^2 n^2 t}]$$

$$\left. - \sum_{n=k+1}^{\infty} a_n \sin \pi nx\, e^{-\pi^2 n^2 t}\right| \leqq \epsilon \sum_{n=1}^{k} e^{-\pi^2 n^2 t_0} + \epsilon^2 + 2\epsilon P + \epsilon$$

$$< \epsilon\,[(1 - e^{-\pi^2 t_0})^{-1} + \epsilon + 2P + 1] \qquad \text{q.e.d.}$$

It is obvious that the assumptions of the lemma are satisfied in this particular case. If $b_n = a_n$

$$|v_{pM}(x_1 0) - u(x_1 0)| = \left|\sum_{n=k+1}^{\infty} a_n \sin \pi nx\right| < \epsilon$$

for $M > M_6$ because of assumption two in the convergence theorem.

[3] The inequality

$$\left|\sum_{n=1}^{k} a_n \sin \pi nx\left[\left(1 - 4r\sin^2\frac{\pi n}{2M}\right)^{tM^2/r} - e^{-\pi^2 n^2 t}\right]\right| \leqq 2\epsilon P$$

follows from the lemma and the fact that because of assumption 2

$$|a_n \sin \pi nx| \leqq |a_n| \leqq 2\int_0^1 |f(x)|\,dx = 2P.$$

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ON THE CONVERGENCE OF APPROXIMATE SOLUTIONS OF THE WAVE EQUATION TO THE EXACT SOLUTION

By Werner Leutert and George G. O'Brien

1. Introduction. In their well known paper, "Uber die Partiellen Differenzengleichungen der Mathematischen Physik," (reference one) Courant, Friedrichs and Lewy prove that if a certain difference equation is substituted for the wave equation

$$\partial^2 u/\partial t^2 = \partial^2 u/\partial x^2$$

the approximate solution, thus obtained, will converge under very general conditions to the solution of the wave equation whenever the mesh size is decreased and the mesh ratio

$$r = \Delta t/\Delta x$$

is chosen less than one and kept constant. They point out that a solution of the difference equation will in general ("in allgemeinen") not converge to the corresponding solutions of the wave equation if the mesh ratio r exceeds one.

In a recent paper by O'Brien, Hyman and Kaplan, "A Study of the Numerical Solution of Partial Differential Equations" (reference two) the above result is quoted and, in addition, it is remarked that if the triangular region of influence of a single point for the difference equation does not include the corresponding region of influence of the point for the differential equation ($r > 1$) convergence is "clearly impossible" and "thus $r \leqq 1$ is a necessary condition for convergence."

In this paper a convergent solution of the difference equation in question for *any positive value of the mesh ratio* r will be established.

2. The problem and its exact solution. The following special problem will be considered. The function $u(x, t)$ satisfies the differential equation

$$\partial^2 u/\partial t^2 = \partial^2 u/\partial x^2 \qquad 0 < x < 1 \qquad 0 < t$$

the boundary conditions

$$u(0, t) = u(1, t) = 0, \qquad t \geqq 0$$

and the initial condition

$$u(x, 0) = f(x) \qquad [\partial u(x, t)/\partial t]_{t=0} \qquad 0 \leqq x \leqq 1 \tag{1}$$

The exact solution of this problem (vibrating string) is well known and given by

$$u(x, t) = \sum_{n=1}^{\infty} a_n \sin \pi n x \cos \pi n t \tag{2}$$

where

$$a_n = 2 \int_0^1 f(x) \sin \pi n x \, dx.$$

The more general case

$$[\partial u(x, t)/\partial t]_{t=0} = g(x), \qquad 0 < x < 1$$

will present no additional difficulties.

It will be assumed that $f(x)$ is continuous in the interval [0, 1] and that it has one sided derivatives at all interior points of that interval. These conditions are sufficient to show that $u(x, t)$ is the unique solution of the problem.

3. The difference equation. The interval $0 \leqq x \leqq 1$ is divided into M subintervals of equal length Δx. The mesh ratio r is defined by the relation $(\Delta x)r = \Delta t$. Let $v(x, t)$ be an exact solution of the difference equation

$$\begin{aligned} v(x, t + \Delta t) + v(x, t - \Delta t) - 2v(x, t) \\ = r[v(x + \Delta x, t) + v(x - \Delta x, t) - 2v(x, t)] \end{aligned} \tag{3}$$

This is the difference equation considered in references one and two. By separation of variables it can easily be found that

$$\begin{aligned} v(x, t) = \sum_{n=1}^{\infty} \sin \pi n x \, [b_n \cos \{(tM/r) \arccos [1 - 2r^2 \sin^2 (\pi n/2M)]\} \\ + c_n \sin \{(tM/r) \arccos [1 - 2r^2 \sin^2 (\pi n/2M)]\}] \end{aligned} \tag{4}$$

is a formal solution of (3) each term of which satisfies (3) and the same boundary conditions as $u(x, t)$, namely

$$v(0, t) = v(1, t) = 0, \qquad t \geqq 0.$$

The b_n and c_n in (4) are arbitrary constants.

4. One exact solution of the difference equation. While it is necessary to know the values of $u(x, 0)$ in (1) for all values of x in the interval $0 < x < 1$, it is sufficient to know the values of $v(x, 0)$ on the mesh points of the first line. These values of $v(x, 0)$ on the mesh points need not coincide with the corresponding values of $u(x, 0)$ on the same mesh points in order to insure convergence.

To any positive integer $M > 1$ we define another positive $k(M)$ such that

$$k(M) < \sqrt[4]{M} \tag{5}$$

and

$$\lim_{M \to \infty} k(M) = \infty \tag{6}$$

i.e. $k(M)$ could be chosen to be the positive integer satisfying the inequality

$$\sqrt[4]{M} - 1 \leqq k(M) < \sqrt[4]{M}.$$

If we put in (4) $c_n = 0$ for all n and $b_n = 0$ for $n > k(M)$ and choose the remaining k different $b_n(M)$ in such a way that to any preassigned $\epsilon > 0$ there exists an $M_1(\epsilon)$ such that for any $M > M_1$

$$| b_n(M) - a_n | < \epsilon/M^{1/4} \text{ uniformly for } n = 1, 2 \cdots k \tag{7}$$

then for any $M > M_1$ the exact solution $v_M(x, t)$ of the difference equation (3) is defined for $0 \leqq x \leqq 1$, $0 \leqq t$ to be

$$v_M(x,t) = \sum_{n=1}^{k(M)} b_n(M) \sin \pi n x \cos [(tM/r) \arccos \{1 - 2r^2 \sin^2 (\pi n/2M)\}]. \quad (8)$$

The initial condition at the line $t = 0$ becomes

$$v_M(x, 0) = \sum_{n=1}^{k} b_n(M) \sin \pi n x. \quad (9)$$

One possible way of satisfying (7) is to choose $b_n = a_n$, $n \equiv 1, 2 \cdots k$ which means that the initial values of $v_M(x, 0)$ are chosen in such away that

$$\int_0^1 [f(x) - v_M(x, 0)]^2 \, dx.$$

is minimized. (Least square error approximation of $f(x)$ by a trigonometric polynomial of k terms.)

5. The convergence theorem. *Assumptions:*

1. $t \geqq 0$, $0 \leqq x \leqq 1$. r and t_0 are fixed positive constants.
2. $\sum_{n=1}^{\infty} a_n \sin \pi n x \equiv f(x)$, $|f(x)| < P$, P independent of x.

$$a_n = 2 \int_0^1 f(x) \sin \pi n x \, dx$$

3. To any $\epsilon > 0$ there exists an $M_1(\epsilon)$ such that for any $M > M_1$

$$|b_n(M) - a_n| < \epsilon M^{-1/4} \text{ for } 1 \leqq n \leqq k(M)$$

4. $k(M) < M^{1/4}$, $\lim_{M \to \infty} k(M) = \infty$.

Theorem: $\lim_{M \to \infty} v_M(x, t) = u(x, t)$ or

$$\lim_{M \to \infty} \sum_{n=1}^{k(M)} b_n(M) \sin \pi n x \cos [(Mt/r) \arccos \{1 - 2r^2 \sin^2 (\pi n/2M)\}]$$

$$= \sum_{n=1}^{\infty} a_n \sin \pi n x \cos \pi n t,$$

r, t, and x are fixed.

6. A Lemma. *Assumptions:*

$$1. \quad r > 0, \qquad 2. \quad 0 \leqq z \leqq \tfrac{1}{2}\pi M^{-1/4}.$$

Lemma: There exists an $M_2(r)$ such that for any $M > M_2$ and any given $\epsilon > 0$

$$|2zr - \arccos (1 - 2r^2 \sin^2 z)| < \epsilon r / M^2.$$

Proof: Since the lemma is obviously satisfied for $r = 1$, choose $r \neq 1$. Choose $M_3(r)$ so that

$$1 - 2r^2 \sin^2 z > 0$$

for all admissible values of z and $M > M_3$. Let

$$F(z) = 2zr - \arccos (1 - 2r^2 \sin^2 z).$$

Then

$$F(0) = 0, \qquad F'(0) = 0$$

and

$$F''(z) = \frac{2r(1 - r^2) \sin z}{(1 - r^2 \sin^2 z)^{3/2}}.$$

A. $r < 1$,

$$0 \leqq F''(z) \leqq 2r \sin z.$$

Integrating twice between the limits zero and z leads to

$$0 \leqq F(z) \leqq 2r(z - \sin z) \leqq (1/3!)2rz^3 \leqq (r\pi^3/3 \cdot 8)M^{-9/4}.$$

B. $r > 1$

$$0 \geqq F''(z) \geqq 2^{5/2} r(1 - r^2) \sin z.$$

In the same way as in case A we obtain

$$0 \geqq F(z) \geqq \frac{2^{5/2} r(1 - r^2)\pi^3}{8 \cdot 6} M^{-9/4}.$$

It is obvious that an $M_2 > M_3$ exists so that for any $M > M_2$ and any positive r

$$| F(z) | < \epsilon r/M^2.$$

7. Proof of the convergence theorem: Putting $z = (\pi n/2M)$ in the lemma of section 6 it follows immediately that there exists an M_2 such that for $M > M_2$

$$|(Mt/r) \arccos \{1 - 2r^2 \sin^2 (\pi n/2M)\} - \pi nt | < \epsilon t/M, \quad n = 1, 2, \cdots k(M)$$

or

$$| \cos [(Mt/r) \arccos \{1 - 2r^2 \sin^2 (\pi n/2M)\}] - \cos \pi nt | < \epsilon t/M. \quad (10)$$

From assumptions 2 and 4 in the convergence theorem it follows immediately that to any $\epsilon > 0$ there exists an $M_4(\epsilon)$ such that for any $M > M_4$ and fixed t

$$\left| \sum_{n=k+1}^{\infty} a_n \sin \pi nx \cos \pi nt \right| < \epsilon. \quad (11)$$

Since for $M > \max(M_1, M_2, M_4)$ (7), (10) and (11) imply

$$|v_M(x,t) - u(x,t)| = \sum_{n=1}^{k} (b_n - a_n) \sin \pi nx \cos \pi nt \sum_{n=1}^{k} (b_n - a_n + a_n) \sin \pi nx$$

$$[\cos\{(tM/r) \arccos(1 - 2r^2 \sin^2(\pi n/2M))\} - \cos \pi nt]$$

$$- \sum_{n=k+1}^{\infty} a_n \sin \pi nx \cos \pi nt | \leqq \epsilon + \epsilon^2 t + \epsilon Pt + \epsilon \leqq \epsilon(2 + \epsilon t + Pt)$$

the convergence theorem is proved.

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REMARKS ON "A STUDY OF THE NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS"

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See this Journal, **29**, pp. 223–251 (1951). The proof of convergence of Δ_R' to D mentioned on page 238 (lines 21, 22) is due to W. Leutert; it will appear in his paper, "On the Convergence of Approximate Solutions of the Heat Equation to the Exact Solution," to be published in the Proceedings of the American Mathematical Society. Lines 18 to 26 on page 228 were inserted by the writer independently of his associate authors.

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